The Khovanov homology of slice disks

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Bryn Mawr College

Princeton Topology Seminar

21 April 2022

Table of Contents

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Motivation for a slice disk

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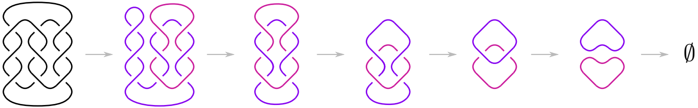

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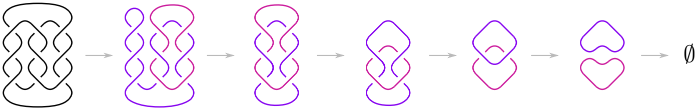

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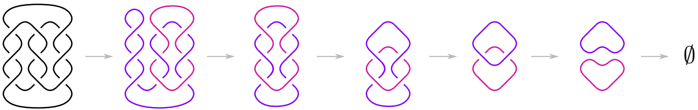

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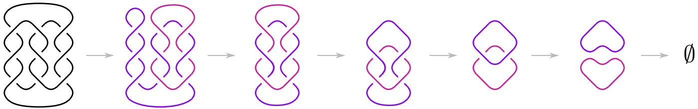

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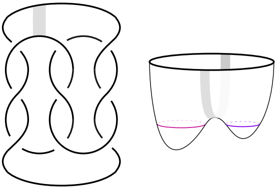

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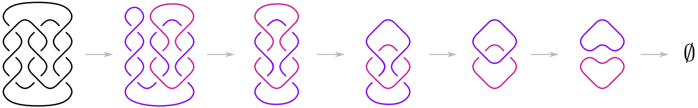

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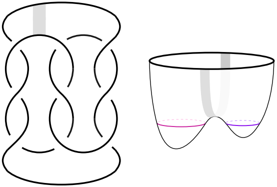

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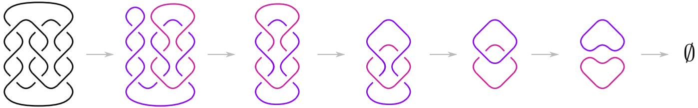

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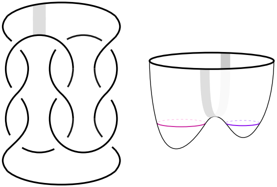

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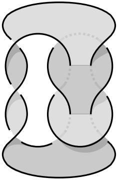

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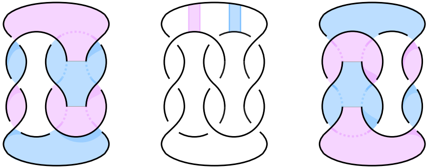

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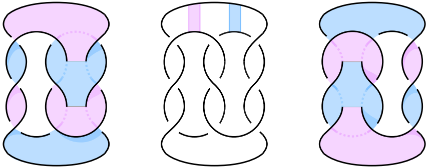

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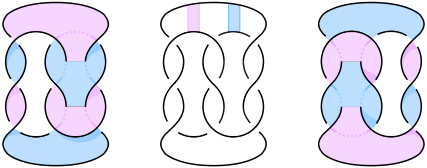

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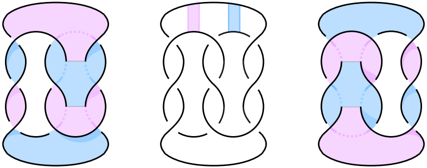

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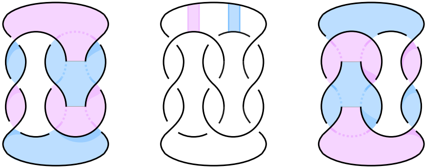

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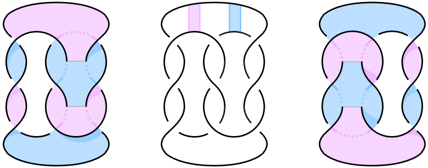

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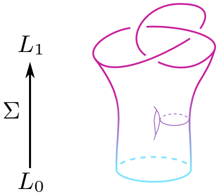

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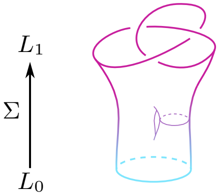

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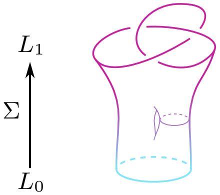

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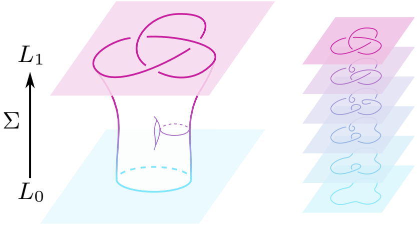

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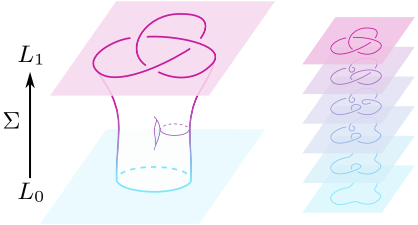

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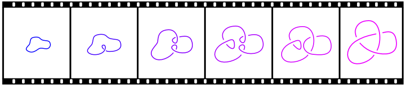

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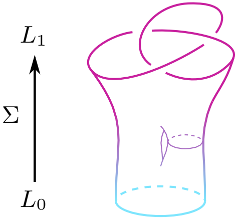			

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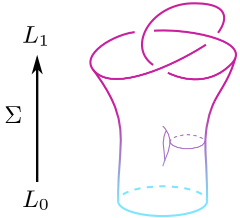			

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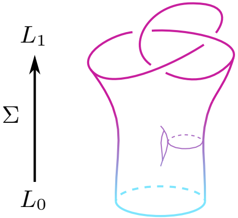		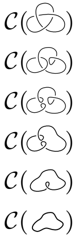	

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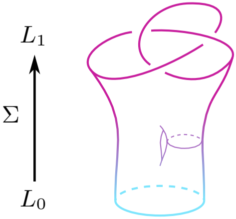		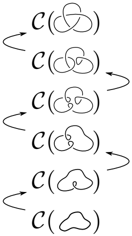	

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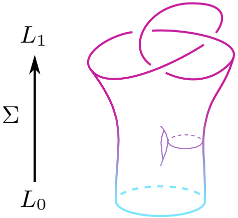		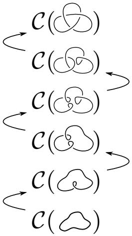	$\mathcal{C}(D_0) \xrightarrow{\mathcal{C}(\Sigma)} \mathcal{C}(D_1)$

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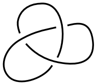

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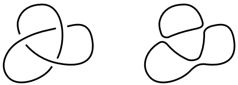

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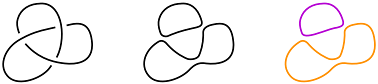

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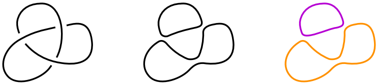

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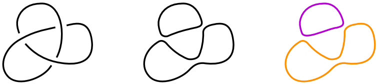

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Let's take a quick look at $\mathcal{C}(3_1)$

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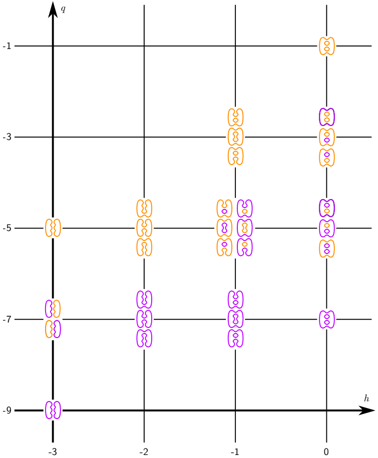

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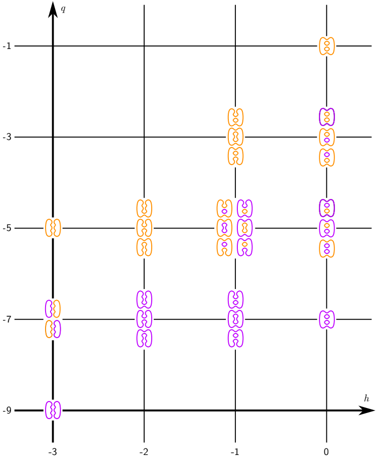

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What do these chain maps $\mathcal{C}(D_{t_i}) \to \mathcal{C}(D_{t_{i+1}})$ look like?

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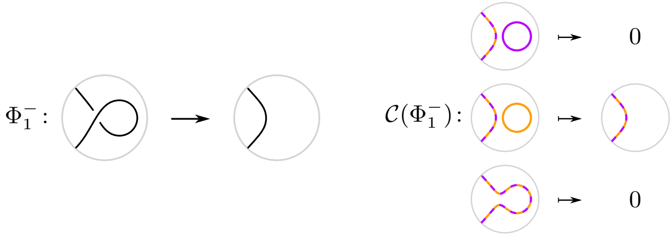

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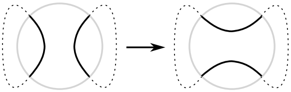

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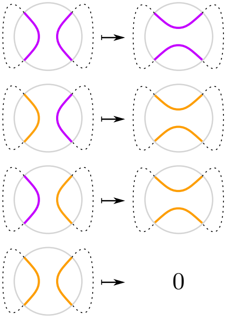

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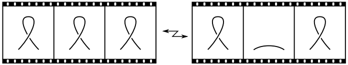

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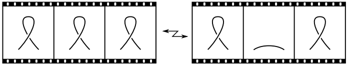

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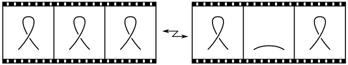

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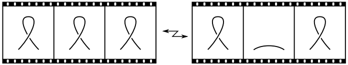

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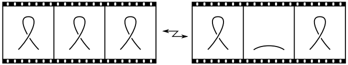

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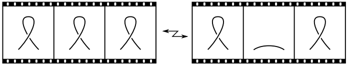

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Invariance can be extended: to link cobordisms in $S^3 \times [0,1]$ and B^4 and to nonorientable cobordisms.

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- conclude Σ, Σ' are not isotopic rel boundary

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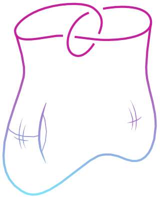

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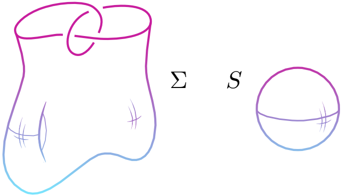

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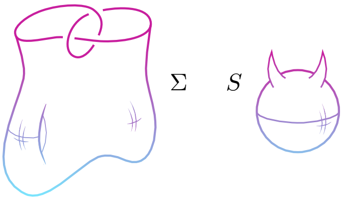

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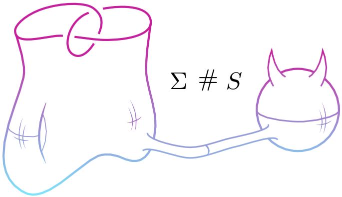

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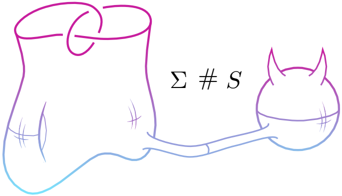

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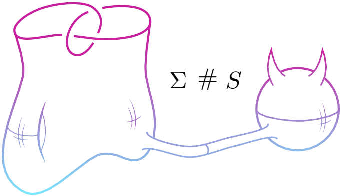
- Then Σ and $\Sigma' := \Sigma \# S$ are (generally) not isotopic rel boundary.



Theorem (Swann, Hayden-Sundberg)

The map on Khovanov homology induced by a link cobordism is invariant under connected sums with knotted 2-spheres.

A brief remark on local knottedness

In general, it is (perhaps too) easy to build such link cobordisms:

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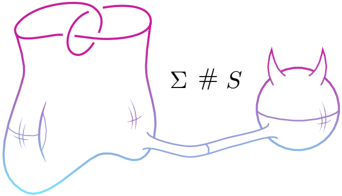

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Takeaway: maps on Khovanov homology detect more than local knotting

Table of Contents

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Question:

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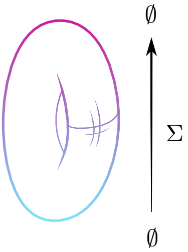

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$$\begin{tikzcd} \emptyset & & \mathbb{Z} \\ \\ \emptyset \arrow[uu, "\Sigma"'] & & \mathbb{Z} \arrow[uu, "\mathcal{H}(\Sigma)"'] \end{tikzcd}$$

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$$\begin{tikzcd} \emptyset & & \mathcal{H}(\Sigma)(1) \in \mathbb{Z} \\ \emptyset \arrow[u, "\Sigma"'] \arrow[r, gray] & {} & 1 \in \mathbb{Z} \arrow[u, "\mathcal{H}(\Sigma)"'] \arrow[u, dotted, bend left] \end{tikzcd}$$

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Lemma

*For a link cobordism $\Sigma\colon \emptyset \to \emptyset$, the φ-**number** of Σ*

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Theorem (Rasmussen, Tanaka)

The φ-numbers associated to connected $\Sigma \subset B^4$ are determined by genus:

- *if* $g(\Sigma) = 1$, *then* $\varphi(\Sigma) = \pm 2$
- *if* $g(\Sigma) \neq 1$, *then* $\varphi(\Sigma) = 0$

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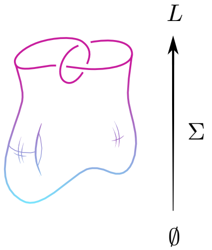

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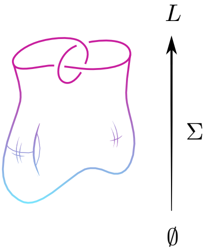

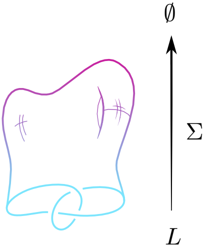

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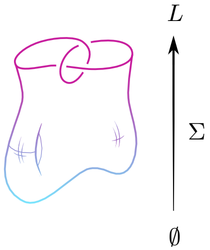
We consider these cases separately in the next two sections.



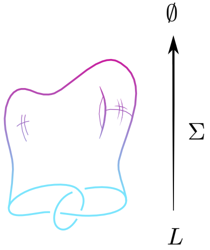


Table of Contents

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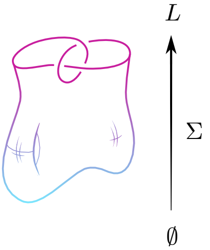

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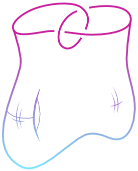

$$\begin{tikzcd} L & & & \mathcal{H}(L) \\ & \arrow[r] & {} & \\ \emptyset \arrow[uu, "\Sigma"'] & & & \mathbb{Z} \arrow[uu, "\mathcal{H}(\Sigma)"'] \end{tikzcd}$$

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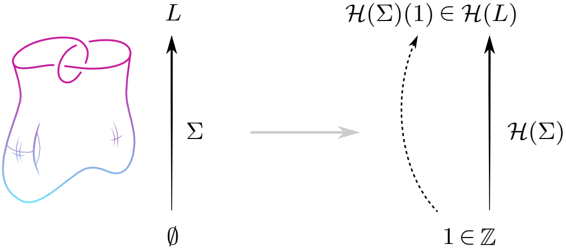

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If so, we say $\Sigma_{0,1}$ are φ-**distinguished**.

Applications of φ-classes

Theorem (Swann, Sundberg)

The slice disks D_ℓ and D_r for 9_{46} are φ-distinguished, and therefore, are not isotopic rel boundary.

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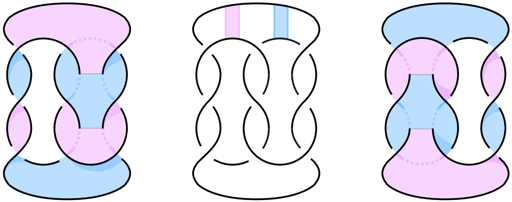

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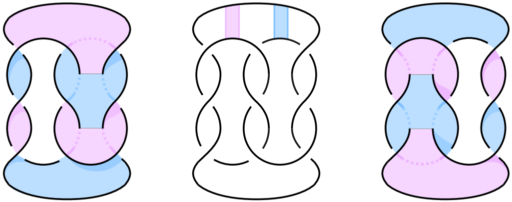

What do $\varphi(D_\ell)$ and $\varphi(D_r)$ look like?

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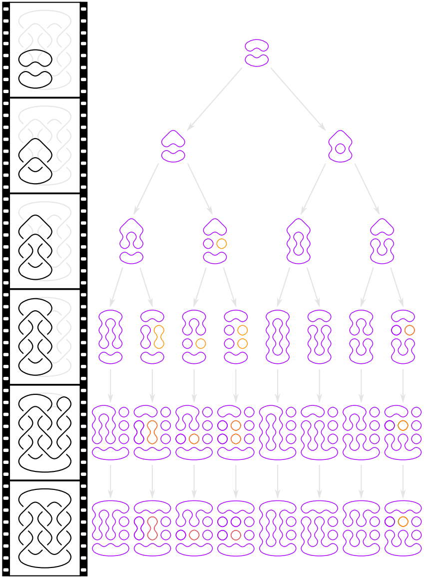

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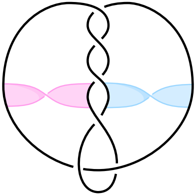

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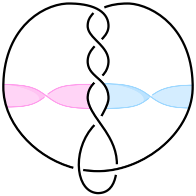

Are there knots with more than 2 unique slice disks?

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The 2^n slice disks bounding $\#_n(9_{46})$ are φ-distinguished, and therefore, are not isotopic rel boundary.

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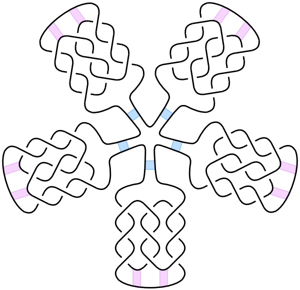
Slice disks are obtained by boundary-summing copies of D_ℓ and D_r.



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- So, extend the 2^n slice disks for $K = \#_n(9_{46})$ by a ribbon-concordance $C\colon K \to K_n$ to a prime knot K_n
- These slice disks are pairwise φ-distinguished using injectivity and functoriality of the induced maps on Khovanov homology:

$$\varphi(C \circ D) = \mathcal{H}(C)(\varphi(D)) \neq \pm\mathcal{H}(C)(\varphi(D')) = \varphi(C \circ D')$$

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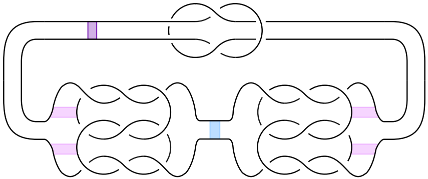

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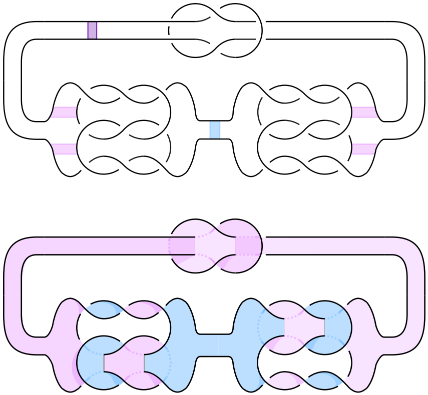

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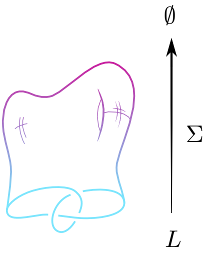

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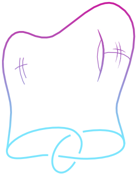

$$\begin{tikzcd} \emptyset & & \mathbb{Z} \\ L \arrow[u, "\Sigma"'] \arrow[r, phantom, "\longrightarrow"] & & \mathcal{H}(L) \arrow[u, "\mathcal{H}(\Sigma)"'] \end{tikzcd}$$

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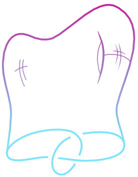
Defining φ^*-numbers

Let's look at the second case: the dual link cobordism $\Sigma\colon L \to \emptyset$.



$$\begin{tikzcd} \emptyset & & \mathcal{H}(\Sigma)(\varphi) \in \mathbb{Z} \\ L \arrow[u, "\Sigma"'] \arrow[r, gray] & & \varphi \in \mathcal{H}(L) \arrow[u, "\mathcal{H}(\Sigma)"'] \arrow[u, dotted, bend left] \end{tikzcd}$$

Method:

- a surface $\Sigma \subset B^4$ with boundary $L \subset S^3$ induces a link cobordism $\Sigma\colon L \to \emptyset$
- it induces a map $\mathcal{H}(\Sigma)\colon \mathcal{H}(L) \to \mathbb{Z}$
- choose a class $\varphi \in \mathcal{H}(L)$, and note that $\mathcal{H}(\Sigma)(\varphi) \in \mathbb{Z}$ is an up-to-sign invariant of the isotopy class of Σ.

Defining φ^*-numbers

Lemma

For a link cobordism $\Sigma\colon L \to \emptyset$ and a class $\varphi \in \mathcal{H}(L)$, the **$\varphi^*$-number**

$$\varphi^*(\Sigma) := \mathcal{H}(\Sigma)(\varphi) \in \mathbb{Z}$$

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Can we find $\Sigma_{0,1} \subset B^4$ bounding a common $L \subset S^3$ *and* a class $\varphi \in \mathcal{H}(L)$ such that $\varphi^*(\Sigma_0) \neq \pm\,\varphi^*(\Sigma_1)$?

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If so, we say $\Sigma_{0,1}$ are $\boldsymbol{\varphi^*}$**-distinguished**.

Applications of φ^*-numbers

Theorem (Hayden-Sundberg)

The pair of slice disks D_ℓ and D_r for the knot K (below) are φ^*-distinguished by the given class $\varphi \in \mathcal{H}(K)$, and therefore, are not isotopic rel boundary.

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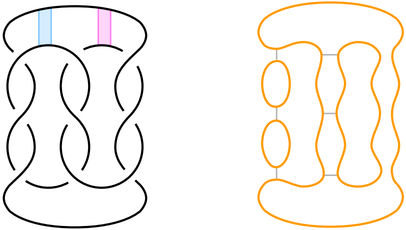

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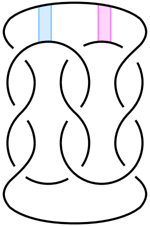

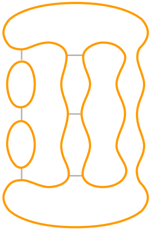

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Proof idea: show $\varphi^*(D_\ell) = 1$ and $\varphi^*(D_r) = 0$

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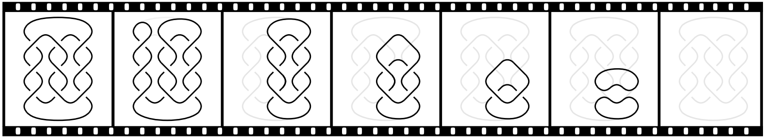

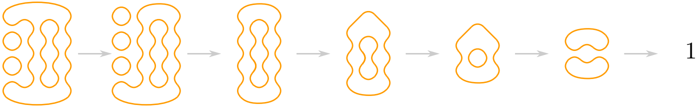

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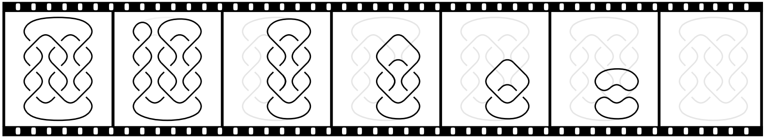

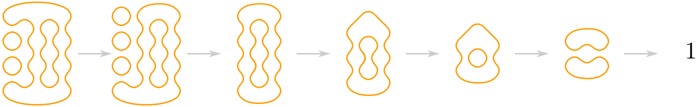

So $\varphi^*(D_\ell) = 1$ and $\varphi^*(D_r) = 0$, as desired.

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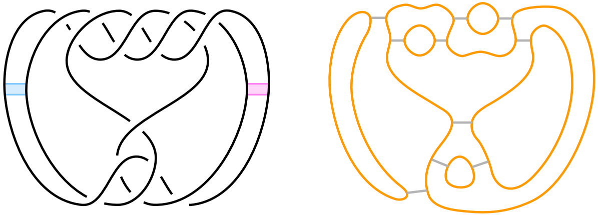

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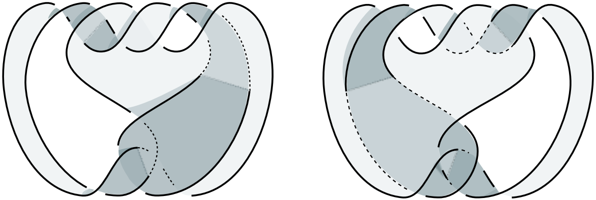

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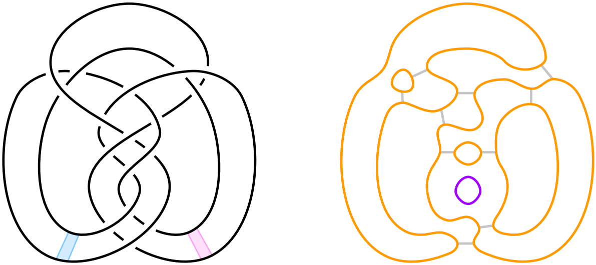

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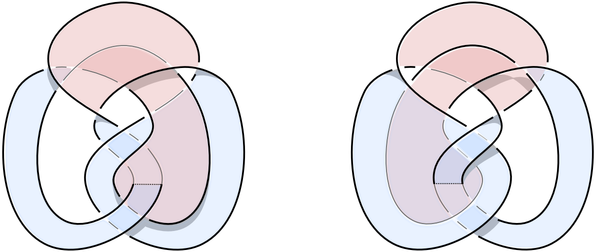

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A pair of surfaces in B^4 are **exotic** if they are topologically isotopic rel boundary, but not smoothly isotopic rel boundary.

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First gauge-theory free proof of exotic surfaces.

Can be extended to higher genus surfaces, asymmetric knots, and ambient isotopy.

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- hard to compute φ-classes

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Thank You!

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