Talk: PACT IV - Khovanov-Jacobsson classes

September 9, 2021 1:35 PM

I MOTIVATION

II KHOVANOV HOMOLOGY OF LINKS

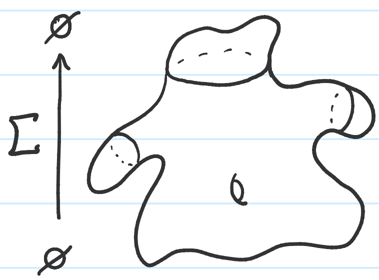
III KHOVANOV HOMOLOGY OF SURFACES

$\boxed{\text{IV. Khovanov homology of closed surfaces}}$

TODAY D means disk

push into $S^3\times[0,1]$

Given knotted surface in B^4, represent it as a link cobordism $\Sigma : \emptyset \to \emptyset$



$$\rightsquigarrow \quad \begin{array}{c} Kh(\emptyset) = \mathbb{Z} \\ \big\uparrow {\scriptstyle Kh(\Sigma)} \\ Kh(\emptyset) = \mathbb{Z} \ni 1 \end{array} \quad \big\uparrow n_\Sigma \quad \rightsquigarrow \quad \boxed{n_\Sigma := |Kh(\Sigma)(1)|} \in \mathbb{Z}$$

Khovanov-Jacobsson number of Σ

NOTE invariant of ambient (no ∂) iso. class of Σ

Ex We calculated:

(a) a std S^2 has $n_{S^2} = 0$

(b) a std T^2 has $n_{T^2} = 2$

Lemma $n_\Sigma \in Kh^{0,\chi(\Sigma)}(\emptyset) = \begin{cases} \mathbb{Z} & \text{if } \chi(\Sigma) = 0 \\ 0 & \text{if } \chi(\Sigma) \neq 0 \end{cases}$

$\Rightarrow$ $n_\Sigma = 0$ for Σ with $\chi(\Sigma) \neq 0$ (i.e. only knotted tori have nontrivial n_Σ)

? Does n_Σ distinguish knotted tori?

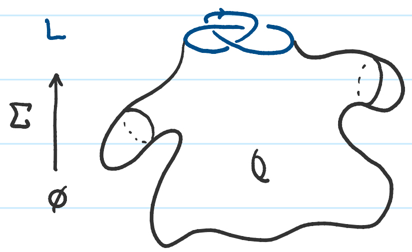
A: no!

Thm (Rasmussen '05, Tanaka '05) Khovanov-Jacobsson numbers are determined by genus: $n_\Sigma = \pm 2$ for $\chi(\Sigma) = 0$ and Σ connected.

acts multiplicatively over multiple components

V Khovanov homology of slice disks

Given a surface* $\Sigma \subset B^4$, represent as a link cob $\Sigma : \emptyset \to L$



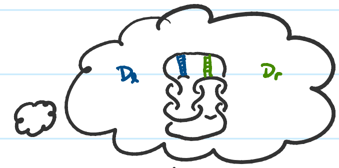
$\rightsquigarrow$ $\mathbb{Z} \to Kh(L)$, $\quad \mathbb{Z} \ni 1 \mapsto KJ_\Sigma$ $\rightsquigarrow$ $\boxed{KJ_\Sigma := |Kh(\Sigma)(1)|} \in Kh(L)$

Khovanov-Jacobsson class of Σ

NOTE Invt of bndry-pres. iso class of Σ

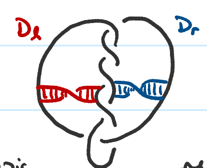
Note: this is a relative version of n_Σ (if $L = \emptyset$ then $n_\Sigma = KJ_\Sigma$)

? Does KJ_Σ distinguish any Σ from any Σ'?
(with $\partial\Sigma = L = \partial\Sigma'$ and $g(\Sigma) = g(\Sigma')$)



Thm 1 (Swann '10, S. '20) The slice disks D_ℓ and D_r induce distinct maps± on Khovanov homology and hence are not isotopic rel bndry.

Thm 2 (S. '20) The slice disks D'_ℓ and D'_r for 6_1,



induce distinct ± maps on Khovanov homology and hence are not isotopic rel bndry.

Thm 3 (S.-Swann '21) The 2^n slices for $\#_n(9_{46})$ idmoKhatanirb

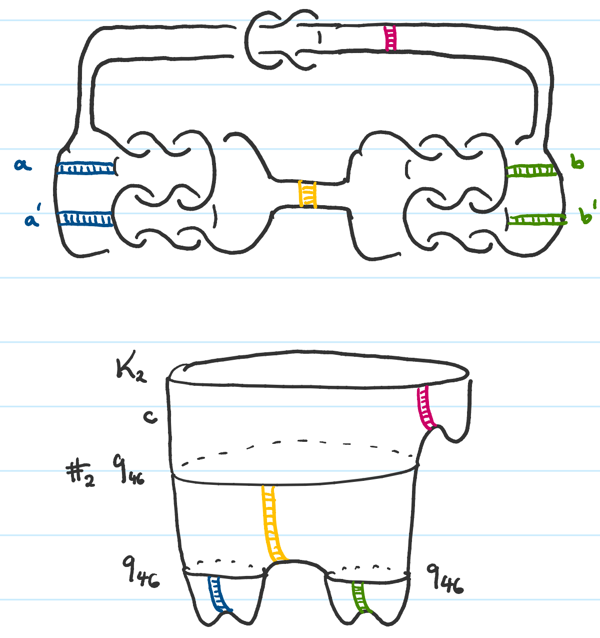
Proof look @ KJ classes ($KJ_{D_\ell} \neq \pm KJ_{D_r}$, etc)

Ex. $\#_2(9_{46}) = 9_{46} \# 9_{46}$ has ___4___ slices, given by

1. either a or a'
2. either b or b'
3. c

(ie pick one of two slices for each copy of 9_{46} and bndry-connect sum)

SIDE BOARD



Note Can also be done for $\#_n(6_1)$ and à-la-carte-sums of 6_1 and 9_{46}

Thm (S.-Swann '21) There are prime knots K_n with 2^n slices that each idmoKhatanirb.

Proof Thm 3 + Thm (L-Z) + Thm (K-L)

can compose link cobs. by stacking

- apply C to $\#_n(9_{46})$ to get K_n
- attach C to any slice D of $\#_n(9_{46})$ to get slice $C \circ D$ of K_n
- KJ classes will remain distinct after attaching C

Thm (Levine-Zemke '19) Ribbon-concordances induce injections on Kh.

(Recall: ribbon-concordances are link cobs $\Sigma \cong S^1 \times [0,1]$ w/o maxima)

Thm (Kirby-Lickorish '79) Every knot is (ribbon) concordant to a prime knot.

(?) How does one show $KJ_\Sigma \neq \pm KJ_{\Sigma'}$?

Note: KJ_Σ is the homology class $\pm[\mathcal{C}KJ_\Sigma]$ of a cycle $\mathcal{C}KJ_\Sigma = \mathcal{C}(\Sigma)(1) \in Kh(L)$

So $0 = KJ_\Sigma \pm KJ_{\Sigma'} = [\mathcal{C}KJ_\Sigma \pm \mathcal{C}KJ_{\Sigma'}]$

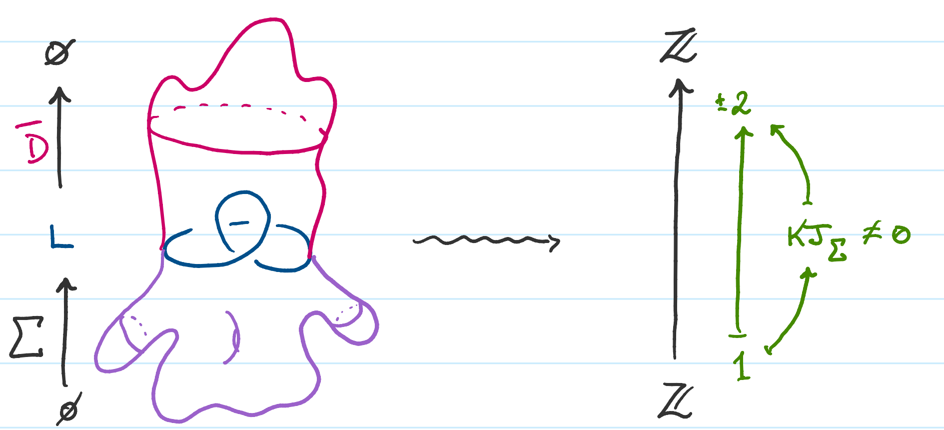
if $c := \mathcal{C}KJ_\Sigma \pm \mathcal{C}KJ_{\Sigma'}$ is a boundary (ie $c = d(e)$ for some $e \in \mathcal{C}Kh^{h=-1}(L)$)

This can be done using a computer!

(?) How do the relative ($L \neq \emptyset$) and absolute ($L = \emptyset$) cases relate?

Thm (Swann '10) Suppose K is slice w/ slice disk D.
Then any $\Sigma : \emptyset \to K$ with $g(\Sigma) = 0$ or 1 has nontrivial KJ_Σ.

Proof ($g = 1$)



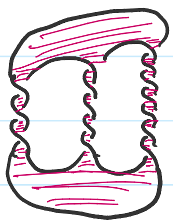
($g=0$) Stack on $\overline{D}\#T^2$ to D (put genus on top)

Same factorization occurs

5

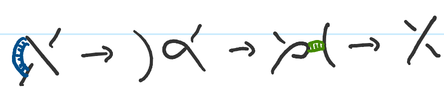
CONVERSE If K bounds a genus 1 surface Σ with $KJ_\Sigma = 0$ then K is <u>not</u> slice.

Thm (Swann '10) The pretzel knots $P(p,q,r)$ with pos. odd p,q,r have genus 1 Seifert surface w/ trivial KJ class and thus are not slice.



$P(3,5,7)$

Conjecture Sliceness of odd 3-stranded pretzel knots is fully determined by KJ-classes.



IDEA The Conway knot has a genus 1 surface (unknotting # 1)

!! nontrivial KJ class