Talk: PACT III - Khovanov homology of surfaces

September 9, 2021 1:35 PM

I MOTIVATION

II KHOVANOV HOMOLOGY OF LINKS

Links ⇝ Diagrams ⇝ Chain cx's ⇝ Homology

L $\qquad D \qquad (\mathcal{C}(D), d) \qquad Kh(D)$

III. KHOVANOV HOMOLOGY OF SURFACES

Goal To a link cob. $\Sigma: L_0 \to L_1$, associate a map $\mathcal{C}(\Sigma): \mathcal{C}(D_0) \to \mathcal{C}(D_1)$

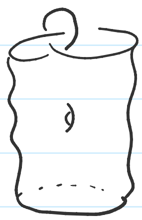
Recall: Σ has a movie $D_0 = D_{t_0}, D_{t_1}, \dots, D_{t_m} = D_1$ DENOTED $D_0 \rightsquigarrow D_1$

($D_{t_i} \longrightarrow D_{t_{i+1}}$ is an isotopy, Reidemeister move, or Morse move)

IDEA

L_1

$\Sigma \uparrow$



L_0

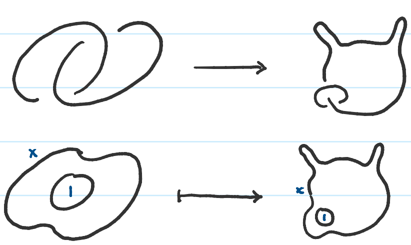
$\xrightarrow{\text{movie}}$

$$\begin{array}{ccc} D_{t_m} & \rightsquigarrow & \mathcal{C}(D_{t_m}) \\ \uparrow & & \uparrow \\ D_{t_{m-1}} & \rightsquigarrow & \mathcal{C}(D_{t_{m-1}}) \\ \uparrow & \text{chain} & \uparrow \\ \vdots & \text{cx's} & \vdots \\ \uparrow & & \uparrow \\ D_{t_1} & \rightsquigarrow & \mathcal{C}(D_{t_1}) \\ \uparrow & & \uparrow \\ D_{t_0} & \rightsquigarrow & \mathcal{C}(D_{t_0}) \end{array} \quad \mathcal{C}(\Sigma)$$

- define a map $\mathcal{C}(D_{t_i}) \longrightarrow \mathcal{C}(D_{t_{i+1}})$ for each of the 3 "moves"
- compose these maps to get $\mathcal{C}(\Sigma)$

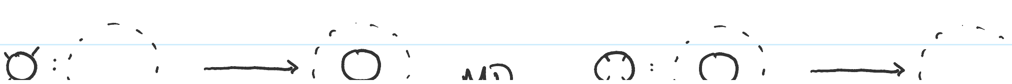
(note: we do so for each α_σ and extend linearly)

① **Isotopy** Given α_σ, apply isotopy to underlying smoothing σ and keep the same labels throughout:



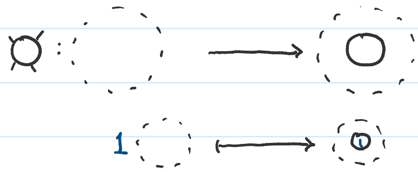
② **Morse** Given α_σ, apply morse move to σ and adjust labels using the following scheme:

MB

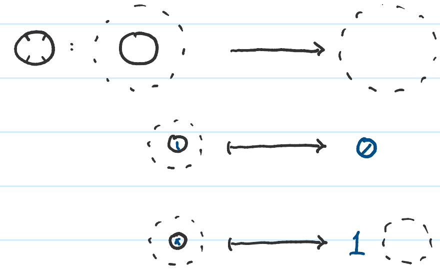


MD

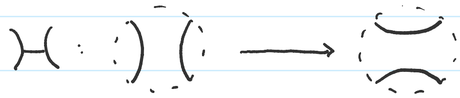
MB.



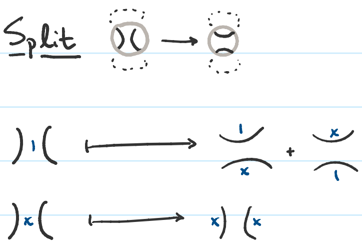
MD



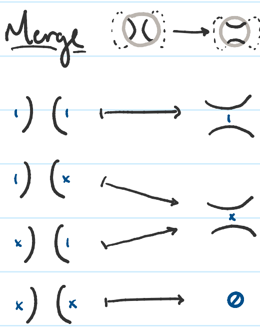
MS



Split

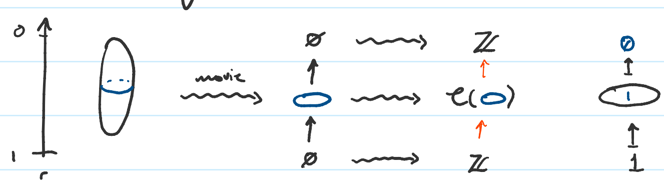


Merge



Example Compute map induced by sphere

- Recall a sphere is a link cob $S : \emptyset \to \emptyset$
- Should induce a map $\mathcal{C}(S) : \mathcal{C}(\emptyset) \to \mathcal{C}(\emptyset)$ ie. $\mathcal{C}(S) : \mathbb{Z} \to \mathbb{Z}$
  $\quad\quad \overset{|||}{\text{0-map}}$
- A movie of S is



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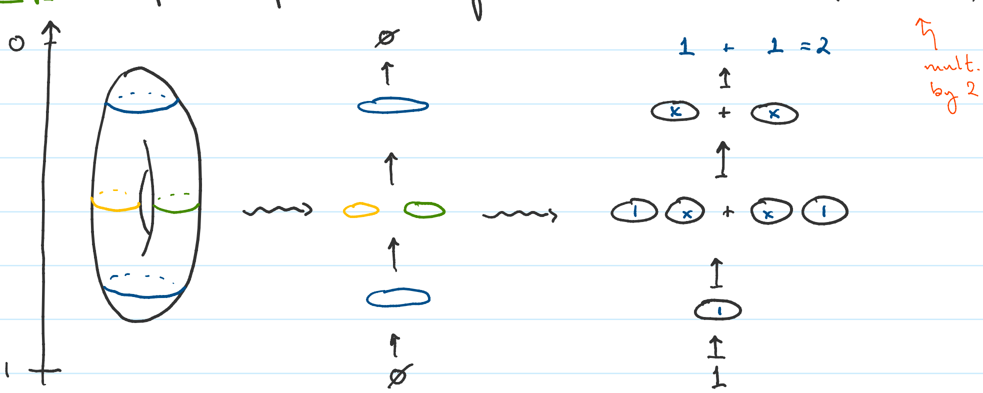
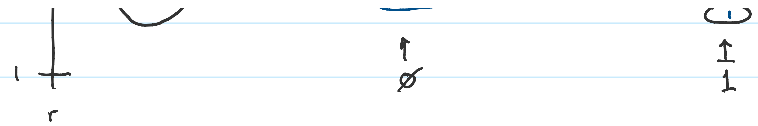
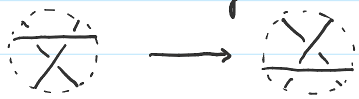
③

Example Compute map induced by torus $T : \emptyset \to \emptyset$ $\quad (\mathcal{C}(T) : \mathbb{Z} \to \mathbb{Z})$





③ Reidemeister: Done similarly, but much harder!

Eg.  has 88 labeled smoothings to be defined on!

Idea: Relate smoothings using Morse moves and apply Morse induced maps to labelings of these smoothings

Thm (Khovanov '00) The Khovanov homology of a link is independent of the chosen diagram.

Proof idea: Reidemeister induced chain maps are chain equivalences and thus induce isomorphisms on homology.

Thm (Khovanov '00) A link cobordism $\Sigma : L_0 \to L_1$ represented by a movie $D_0 \to D_1$ induces a chain map $\mathcal{C}(\Sigma) : \mathcal{C}(D_0) \to \mathcal{C}(D_1)$ with induced map on homology $Kh(\Sigma) : Kh(D_0) \longrightarrow Kh(D_1)$.

⚠ Only useful if invariant under isotopy of Σ (ie. movie doesn't matter)

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④

Thm (Jacobsson '04, BN '05, K '06) The map $Kh(\Sigma)$ is invariant, up to sign, under bndry-pres. isotopy of Σ.

Proof idea: Every movie is related by a sequence of "movie moves" (CARTER-SAITO-SATOH)

And maps induced by related movie moves are chain htpc. up to sign

TAKEAWAY: contrapositive $\boxed{Kh(\Sigma) \neq \pm Kh(\Sigma') \Rightarrow \Sigma \neq \Sigma'}$

Exercise. Show $\mathcal{C}(\Sigma) : \mathcal{C}^{i,j}(D_0) \longrightarrow \mathcal{C}^{i, j+\chi(\Sigma)}(D_1)$
(same will hold for homology)

Exercise. Show $C(\Sigma) : C(D_0) \longrightarrow C(D_1)$

(same will hold for homology)