Talk: PACT II - Khovanov homology of links

September 9, 2021 1:35 PM

I MOTIVATION

II. Khovanov Homology

IDEA Khov. hom. is a functor:

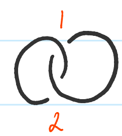
TOP				ALG
Links L	$\rightsquigarrow$ diagrams D	$\rightsquigarrow$ chain cx's $C(D)$	$\rightsquigarrow$	$\mathbb{Z}$-modules $Kh(L)$ (groups)
Link cobs $\Sigma: L_0 \to L_1$	$\rightsquigarrow$ movies $D_0, \dots, D_1$	$\rightsquigarrow$ chain maps $C(\Sigma): C(D_0) \to C(D_1)$	$\rightsquigarrow$	$\mathbb{Z}$-linear maps $Kh(\Sigma): Kh(L_0) \to Kh(L_1)$

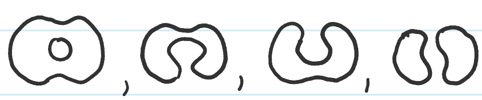
↑ make sure independent of these choices

a. Homology groups

Defn A crossing  in a link diagram D can be smoothed in two ways: as a 0-smoothing  or a 1-smoothing 

A smoothing of D is a planar 1-mfld where every crossing has been smoothed.

Ex. The smoothings of the Hopf link diagram $D =$  are



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In general, a diagram with n-crossings has 2^n smoothings

Note If we enumerate the crossings of D, then a smoothing is defined by a binary sequence $\sigma = (\sigma_1, \dots, \sigma_n)$ where the i^{th} bit indicates that the i^{th} crossing is σ_i-smoothed

Defn A **labeled smoothing** α_σ is a smoothing σ where each component is labeled with a 1 or an x. ← based off of underlying TQFT: using $\mathbb{Z}[x]/(x^2)$

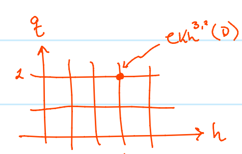
Ex The labeled smoothings α_σ for $\sigma =$ ⓞ are

In general, a smoothing with m components has $\underline{2^m}$ labeled smoothings.
There are $\underline{12}$ labeled smoothings for ⓞ

Defn Let D be a diagram representing an oriented link L with enumerated crossings. The **Khovanov chain complex** $\mathcal{C}(D)$ is the $\mathbb{Z}$-module generated by the labeled smoothings of D.
(group; generally, an R-module)

Ex. $2\,ⓞ^{x} - 7\,()^{\,} \in \mathcal{C}(ⓞ)$ (Note $\emptyset \in \mathcal{C}(ⓞ)$)

Gradings If D has n crossings (n_+ positive, n_- negative) we set



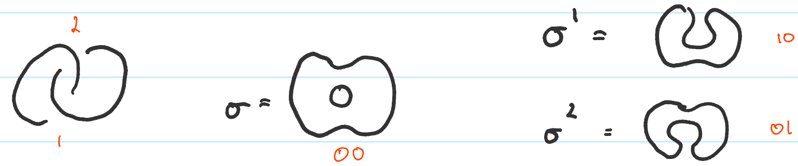
- $h(\alpha_\sigma) = (\#\ \text{1-smoothings in } \sigma) - n_-$ (homological)
- $q(\alpha_\sigma) = v_1(\alpha_\sigma) - v_x(\alpha_\sigma) + h(\alpha_\sigma) + n_+ - n_-$ (quantum)

where $v_1(\alpha_\sigma) = (\#\ \text{1-labels on } \alpha_\sigma)$ and similarly for $v_x(\alpha_\sigma)$

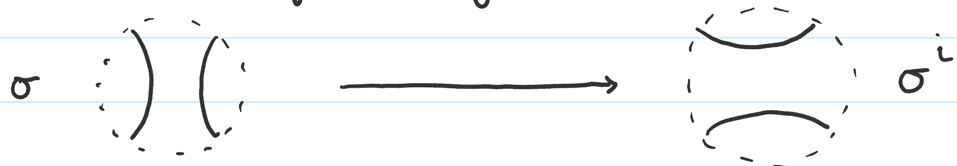
Denote by $\mathcal{C}Kh^{h,q}(D)$

<u>Differential</u> A differential $d: \mathcal{C}^h(D) \longrightarrow \mathcal{C}^{h+1}(D)$ is defined on each α_σ and extended linearly. Define $d(\alpha_\sigma)$ by the following process:

- let $\sigma^i = (\sigma_1, \dots, \sigma_{i-1}, 1, \sigma_{i+1}, \dots, \sigma_n)$

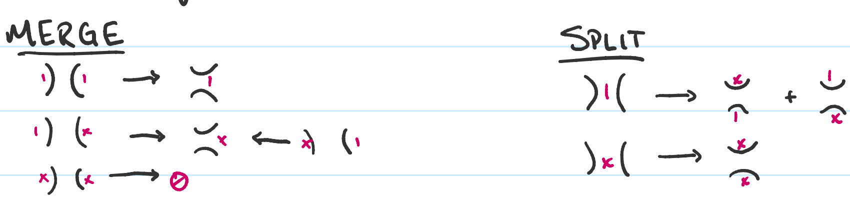


- note σ and σ^i are the same planar 1-mfld except @ the i^{th} crossing; locally at $\times$ we have



and this change either splits or merges connected components of σ

- let α_{σ^i} be the labeled smoothing obtained from the following local label changes on α_σ



- let $\xi^i = \sum_{j<i} \sigma_j$

- define $d(\alpha_\sigma) = \sum_{\{i \mid \sigma_i = 0\}} (-1)^{\xi^i} \alpha_{\sigma^i}$

Note $h(\alpha_{\sigma^i}) = h(\alpha_\sigma) + 1 \implies d^h : \mathcal{C}^h(D) \longrightarrow \mathcal{C}^{h+1}(D)$

(cohomology thing)

Fact $d^2 = 0$ (ξ^i make this happen)

Defn The homology groups $Kh(D)$ associated to $(\mathcal{C}(D), d)$ is called the Khovanov homology of D.

- The isomorphism class of $Kh(D)$ is independent of the chosen enumeration of D as well as the diagram D.
- Categorifies the Jones polynomial

FACT $\mathcal{C}Kh(\emptyset) = \mathbb{Z}$ and $Kh(\emptyset) = \mathbb{Z}$