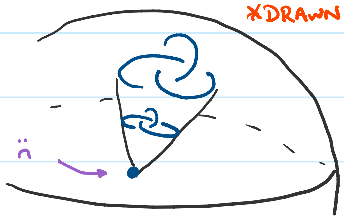
I. Motivation

Classic Question: Given a knot $K \subset S^3$, is there a disk $D \subset B^4$ w/ $\partial D = K$?

DRAWN DOWN A DIM



$S^3 = \mathbb{R}^3 \cup \{\infty\}$

$B^4 = S^3 \times [0,1] / S^3 \times \{0\}$ ⟵ gives a way of discussing radius (x, r)

NOTE: $S^3 = \partial B^4$

Yes, there is always a top disk ($K \times I / K \times \{0\} \cong D^2$)

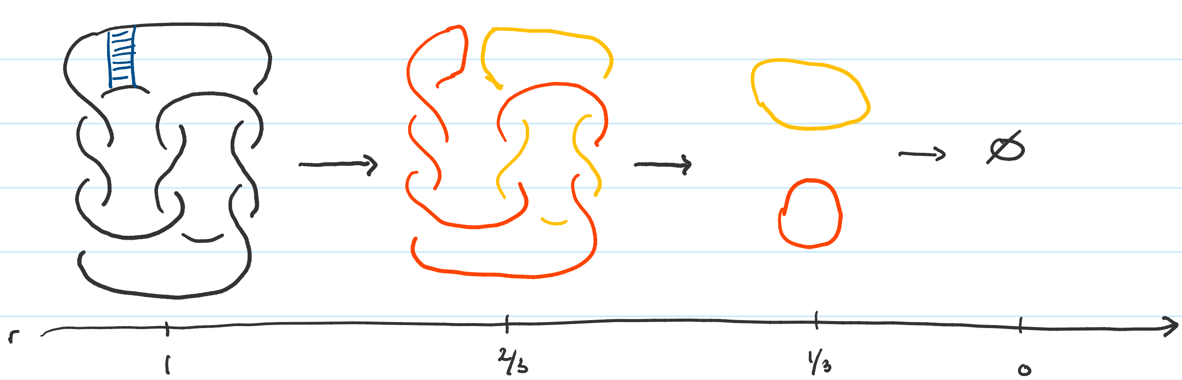
Not necessarily a smooth disk

Defn If $\exists$ smooth D with $\partial D = K$, then we say K is *slice* and that D is a *slice disk*.

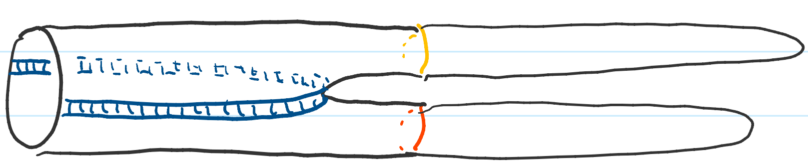
(!) Some knots are not slice (e.g. ) and some are:

Example $K = 9_{46}$ is slice

To see this, we view the $\underbrace{\text{level sets}}_{\text{links}}$ of the disk w.r.t. radius r



(Morse) saddle →



Smooth disk!

Can view this slice in S^3 (with singularities!) by collapsing r

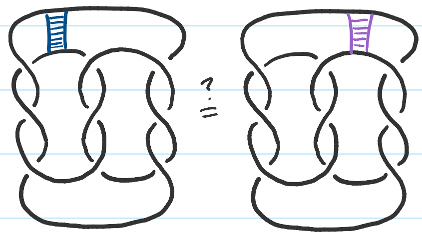
Above question becomes: which knots are slice ? *EXISTENCE QUESTION*

↳ well-studied: known up to 10 crossings

Follow-up Question: When are two slice disks equivalent? *UNIQUENESS QUESTION*

① There is a second slice for 9_{46}!

Let D_ℓ and D_r be these slices:



Are D_ℓ and D_r equivalent?

↳ Yes, ambiently by a 180° rotation → true of many slices!

↳ Not necessarily rel ∂...

Above question becomes: when are two slices related by a boundary-preserving isotopy?

We need a way of distinguishing slice disks:

- look @ $\pi_1(B^4 - D_{\ell,r})$
- gauge theory
- Alexander modules (Miller-Powell)
- knot Floer homology (Juhasz-Zemke)

I use Khovanov homology

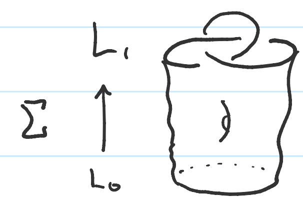
II. TOOLS & SUCH

① LINK COBORDISMS

Defn A *link cobordism* $\Sigma \subset \mathbb{R}^3 \times [0,1]$ is a smooth, compact, oriented, properly-embedded surface.

Note $\partial\Sigma$ is a pair of oriented links $L_i = \Sigma \cap (\mathbb{R}^3 \times \{i\})$, $i \in \{0,1\}$

We write $\Sigma : L_0 \to L_1$ or $L_0 \to L_1$



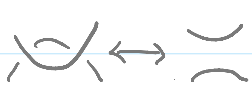
Some ex's:

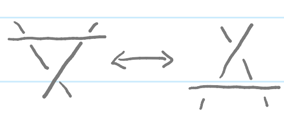
- Slice disks ($\emptyset \to K$ or $K \to \emptyset$; $\Sigma \cong D^2$)
- Closed surfaces in B^4 ($\emptyset \to \emptyset$)
- Seifert surfaces pushed into B^4 ($\emptyset \to K$ or $K \to \emptyset$; $\Sigma \hookrightarrow S^3$)
- Concordances ($\Sigma \cong S^1 \times [0,1]$)

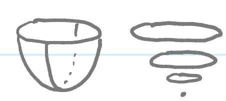
How do we study links? With diagrams!
How do we study link cobordisms? With sequences of diagrams!

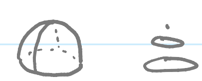
Defn A *movie* of a link cobordism $\Sigma : L_0 \to L_1$ is a finite sequence of diagrams with each successive pair related by a planar isotopy, Reidemeister move, or Morse move.

R1. 

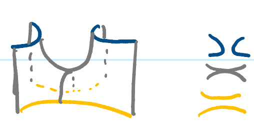
R2. 

R3. 

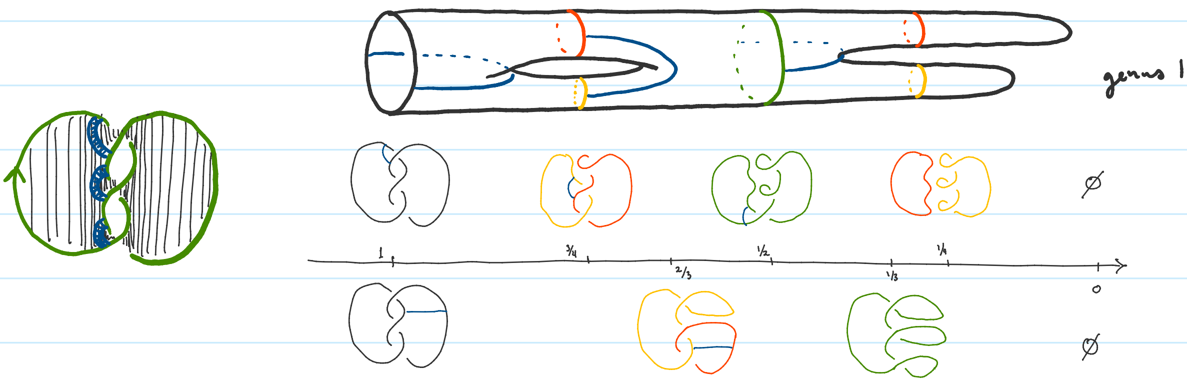
MB. $\emptyset \to \bigcirc$ 

MD. $\bigcirc \to \emptyset$ 

MS. 



GROUP EXERCISE Try to find a movie describing the surface + determine genus



⚠ **NOT UNIQUE!** Isotopic cobordisms induce different movies!