The Khovanov homology of slice disks

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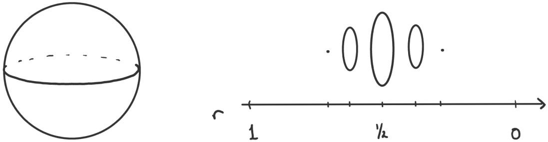

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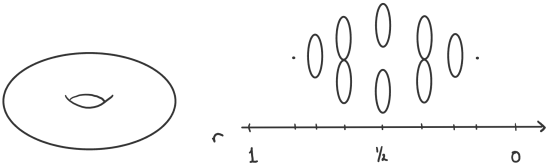

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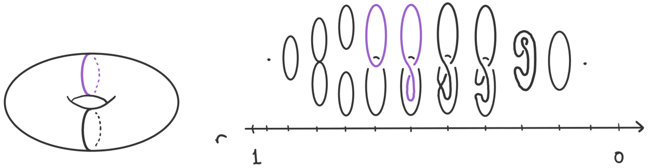

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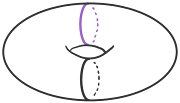

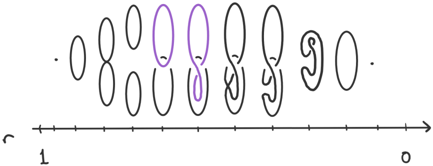

Takeaway: We can answer this question by describing the level sets of a disk D.

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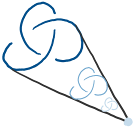

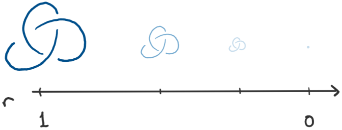

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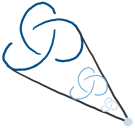

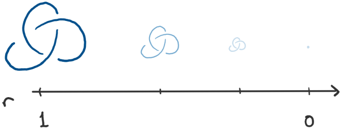

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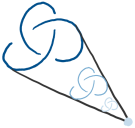

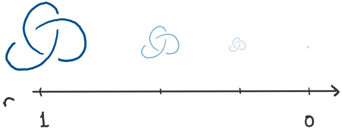

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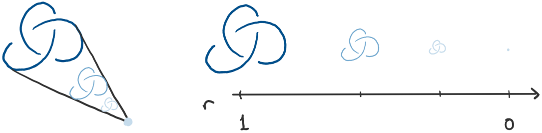

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Definition

A knot $K \subset S^3$ that bounds a smooth, properly embedded disk $D \subset B^4$ is a **slice knot** and D is a **slice disk**.

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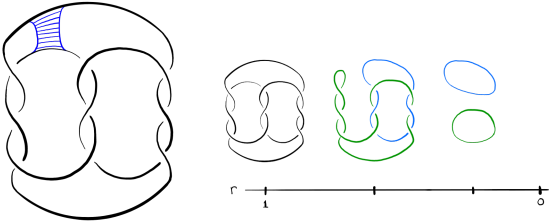

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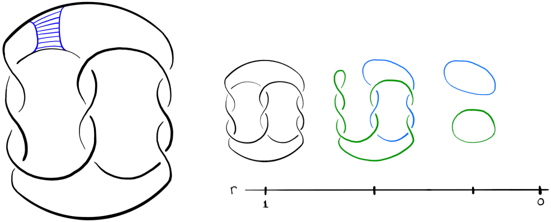

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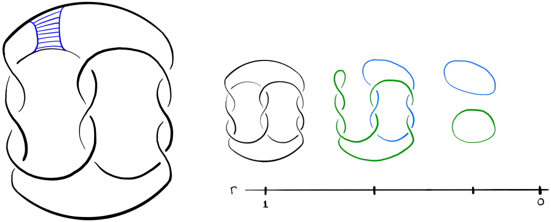

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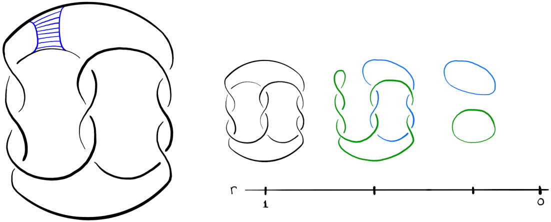

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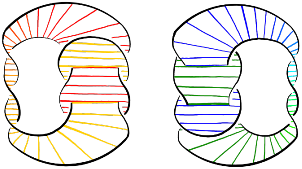

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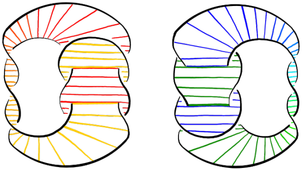

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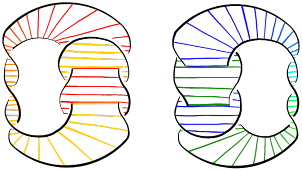

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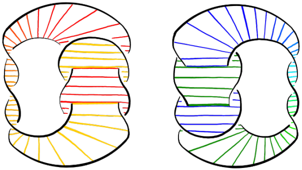

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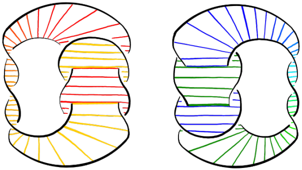

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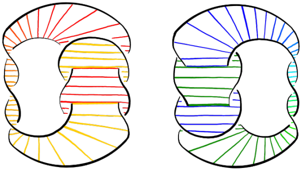

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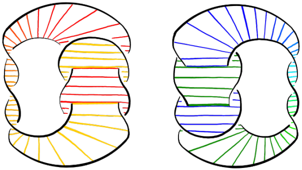

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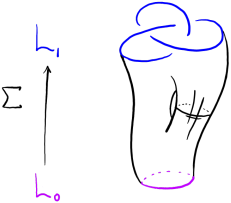

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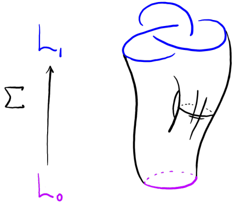

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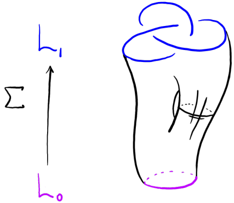

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Definition. A link cobordism $\Sigma\colon L_0 \to L_1$ can be represented as a **movie**: a finite sequence of diagrams $\{D_{t_i}\}_{i=0}^{n}$, with each successive pair related by an isotopy, Morse move, or Reidemeister move.

Idea of Khovanov homology

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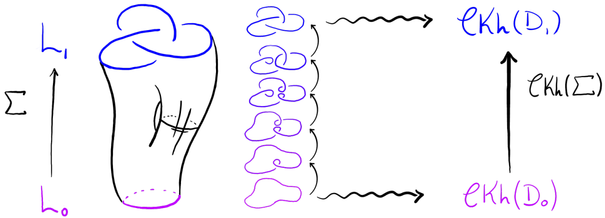

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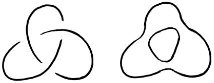

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Theorem (Khovanov '00)

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- The diagrams D_{t_i} in the movie each have an associated chain complex

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Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^n$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

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with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

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- But they have one very useful property!

Invariance

Theorem (Jacobsson '04, Bar-Natan '05, Khovanov '06)

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$$\mathsf{Kh}(\Sigma) \neq \pm\mathsf{Kh}(\Sigma') \implies \Sigma \not\simeq_\partial \Sigma'$$

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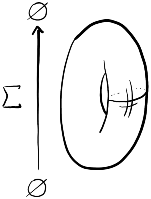

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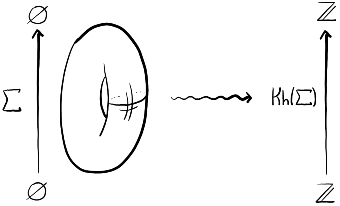

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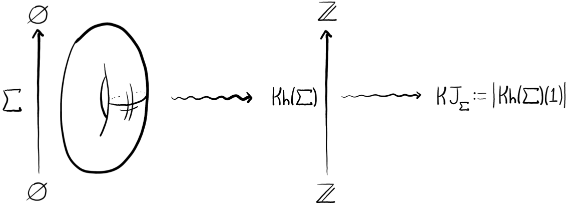

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- This map is determined by $\mathsf{Kh}(\Sigma)(1) \in \mathbb{Z}$, so this integer is an up-to-sign invariant of the (ambient) isotopy class of Σ

Rasmussen-Tanaka

Lemma

For a link cobordism $\Sigma\colon \emptyset \to \emptyset$, the Khovanov-Jacobsson number

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Theorem (Rasmussen '05, Tanaka '05)

Khovanov-Jacobsson numbers of connected Σ are determined by genus:

- *if $g(\Sigma) = 1$, then $\mathsf{KJ}_\Sigma = 2$*
- *if $g(\Sigma) \neq 1$, then $\mathsf{KJ}_\Sigma = 0$*

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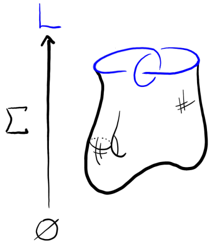

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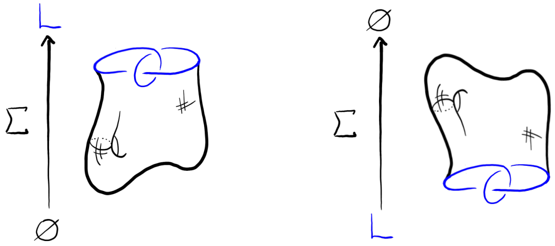

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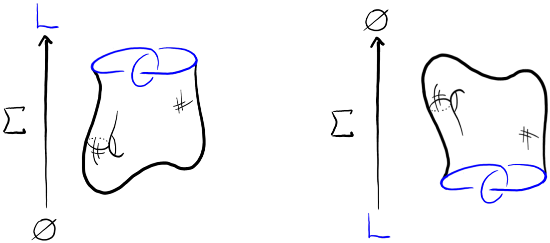
We consider these cases separately.



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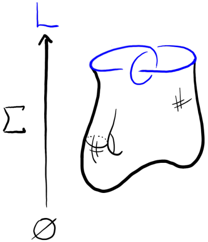

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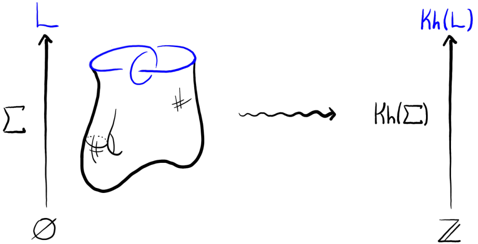

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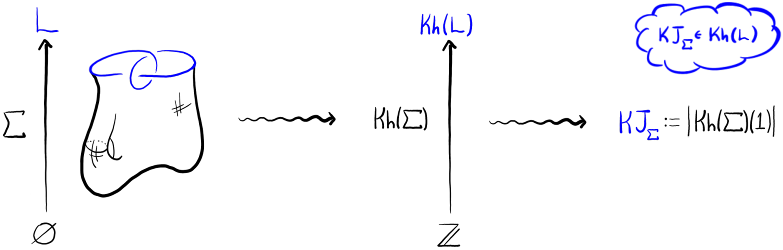

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This map is determined by $\mathsf{Kh}(\Sigma)(1) \in \mathsf{Kh}(L)$, so this homology class is an up-to-sign invariant of the (relative) isotopy class of Σ.

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For a link cobordism $\Sigma\colon \emptyset \to L$, the Khovanov-Jacobsson class

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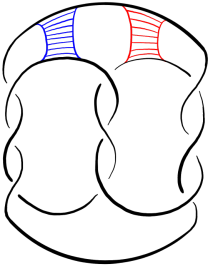
Theorem (Swann '10, S. '20)

The slice disks D_ℓ and D_r for 9_{46} have distinct Khovanov-Jacobsson classes $\mathsf{KJ}_{D_\ell} \neq \mathsf{KJ}_{D_r}$, and therefore, are not isotopic rel boundary.



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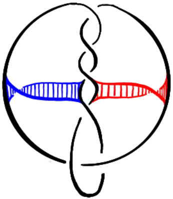

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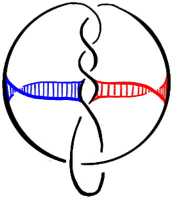
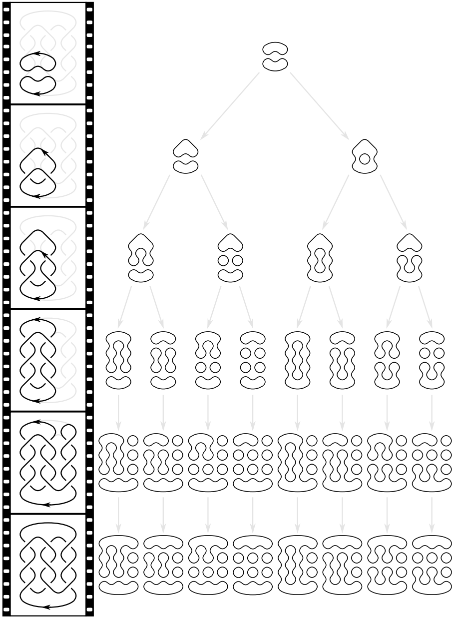


Note: this uniqueness is also known through other techniques.

Calculation for 9_{46}



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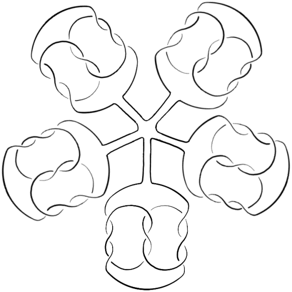

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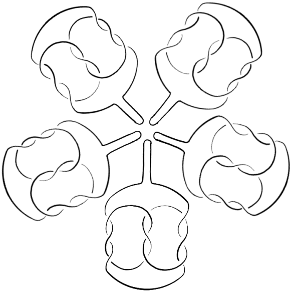

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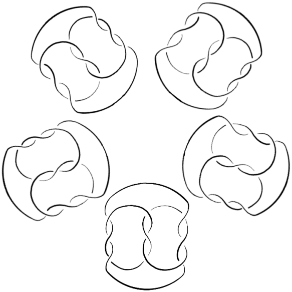

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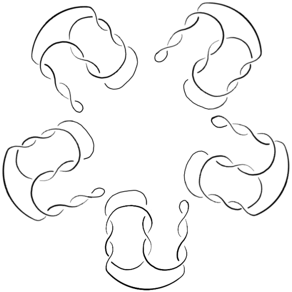

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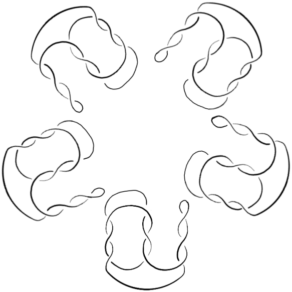

This can also be done with $\#_n(6_1)$, or even by using combinations of 9_{46} and 6_1.

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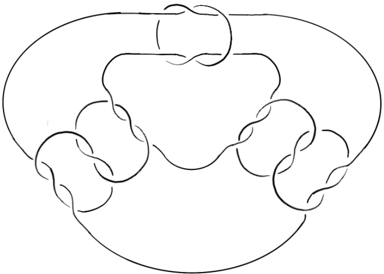

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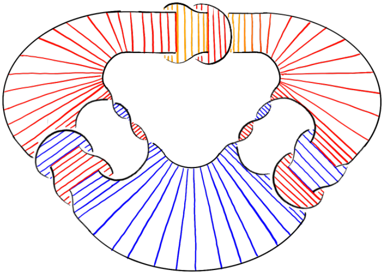

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Theorem (Swann '10)

If $\Sigma\colon \emptyset \to K$ *has genus* $g(\Sigma) = 1$ *and* $\mathsf{KJ}_\Sigma = 0$ *then* K *is not slice.*

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Are we still on case 1?

Downside to Khovanov-Jacobsson classes:

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Is there a better way?

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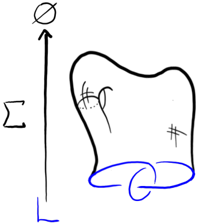
Reverse cobordism

Case 2: Consider a link cobordism $\Sigma\colon L \to \emptyset$



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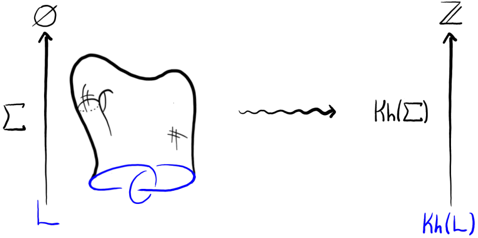

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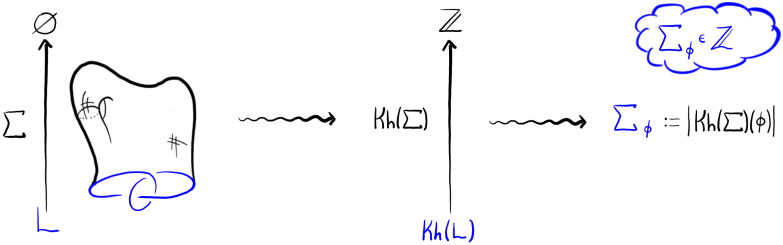

Consider the induced map $\mathsf{Kh}(\Sigma)\colon \mathsf{Kh}(L) \to \mathbb{Z}$

Choose a class $\phi \in \mathsf{Kh}(L)$, and note that $\mathsf{Kh}(\Sigma)(\phi) \in \mathbb{Z}$ is an up-to-sign invariant of the (relative) isotopy class of Σ.

Reverse cobordism

Case 2: Consider a link cobordism $\Sigma\colon L \to \emptyset$

Lemma

For a link cobordism $\Sigma\colon L \to \emptyset$ *and a class* $\phi \in \mathsf{Kh}(L)$*, the integer*

$$\Sigma_\phi := |\mathsf{Kh}(\Sigma)(\phi)| \in \mathbb{Z}$$

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Questions:
Do these invariants distinguish any surfaces?
Are they better than Khovanov-Jacobsson classes?

Quick results

Theorem (Hayden-S. '21)

The pair of slice disks D_ℓ and D_r for the knot K (below) induce distinct maps on Khovanov homology, distinguished by the given class $\phi \in \mathsf{Kh}(K)$, and therefore, are not isotopic rel boundary.

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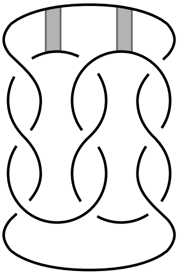

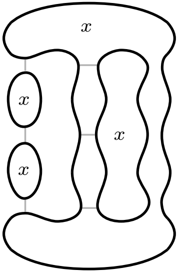

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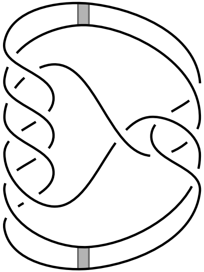

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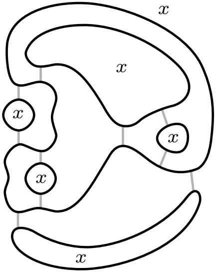

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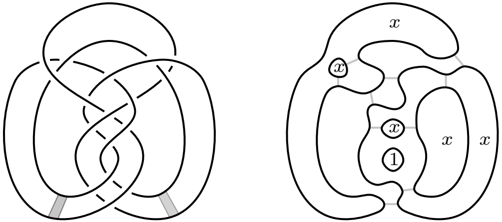

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Exotic slices

Fact:
The slices for 6_1, 9_{46}, and $15n_{103488}$ are not even topologically isotopic rel boundary.

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A pair of slice disks are *exotic* if they are topologically isotopic rel boundary, but not smoothly isotopic rel boundary.

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The slices for $17nh_{74}$ *are topologically isotopic rel boundary.*

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The induced maps on Khovanov homology detect exotic pairs of slice disks.

Can be extended to an infinite family of knots bounding pairs of ambiently non-isotopic surfaces of any genus.

Comparisons

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Case 2:

- By choosing ϕ wisely, it is easier to compute Σ_ϕ
- Comparing integers is easy

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Thank You!

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