The Khovanov homology of slice disks

Isaac Sundberg
Collaborators: Jonah Swann & Kyle Hayden

Bryn Mawr College

Duke Geometry & Topology Seminar

24 January 2022

Table of Contents

1. Motivation
2. Khovanov homology of surfaces
3. Khovanov homology of knotted surfaces
4. Khovanov homology of slice disks: Khovanov-Jacobsson classes
5. Khovanov homology of slice disks: reverse cobordisms
6. Future work

Table of Contents

1. Motivation
2. Khovanov homology of surfaces
3. Khovanov homology of knotted surfaces
4. Khovanov homology of slice disks: Khovanov-Jacobsson classes
5. Khovanov homology of slice disks: reverse cobordisms
6. Future work

Motivation for slice disks

Question:

Motivation for slice disks

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Motivation for slice disks

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Recall: We can view the 3-sphere and 4-ball as follows:

- $S^3 = \mathbb{R}^3 \cup \{\infty\}$
- $B^4 = S^3 \times [0,1]/S^3 \times \{0\}$

Motivation for slice disks

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Recall: We can view the 3-sphere and 4-ball as follows:

- $S^3 = \mathbb{R}^3 \cup \{\infty\}$
- $B^4 = S^3 \times [0,1]/S^3 \times \{0\}$

This allows us to view surfaces $F \subset B^4$ by their level sets $F_i = F \cap (S^3 \times \{i\})$.

Motivation for slice disks

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Recall: We can view the 3-sphere and 4-ball as follows:

- $S^3 = \mathbb{R}^3 \cup \{\infty\}$
- $B^4 = S^3 \times [0,1] / S^3 \times \{0\}$

This allows us to view surfaces $F \subset B^4$ by their level sets $F_i = F \cap (S^3 \times \{i\})$.

Example:

Motivation for slice disks

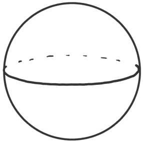
Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

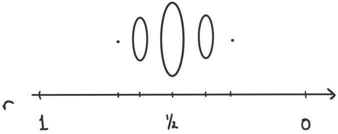
Recall: We can view the 3-sphere and 4-ball as follows:

- $S^3 = \mathbb{R}^3 \cup \{\infty\}$
- $B^4 = S^3 \times [0,1]/S^3 \times \{0\}$

This allows us to view surfaces $F \subset B^4$ by their level sets $F_i = F \cap (S^3 \times \{i\})$.

Example: A sphere in the 4-ball might look like:





Motivation for slice disks

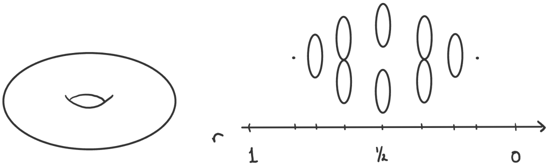
Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Recall: We can view the 3-sphere and 4-ball as follows:

- $S^3 = \mathbb{R}^3 \cup \{\infty\}$
- $B^4 = S^3 \times [0,1]/S^3 \times \{0\}$

This allows us to view surfaces $F \subset B^4$ by their level sets $F_i = F \cap (S^3 \times \{i\})$.

Example: A torus in the 4-ball might look like:



Motivation for slice disks

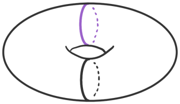
Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

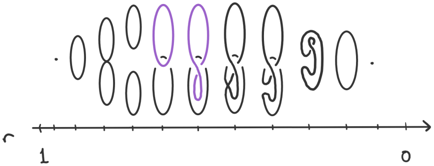
Recall: We can view the 3-sphere and 4-ball as follows:

- $S^3 = \mathbb{R}^3 \cup \{\infty\}$
- $B^4 = S^3 \times [0,1]/S^3 \times \{0\}$

This allows us to view surfaces $F \subset B^4$ by their level sets $F_i = F \cap (S^3 \times \{i\})$.

Example: A torus in the 4-ball might look like:





Motivation for slice disks

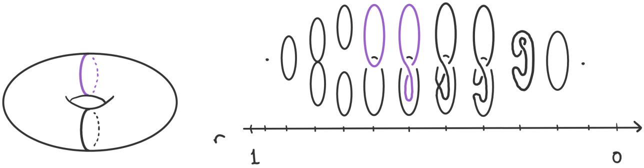
Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Recall: We can view the 3-sphere and 4-ball as follows:

- $S^3 = \mathbb{R}^3 \cup \{\infty\}$
- $B^4 = S^3 \times [0,1]/S^3 \times \{0\}$

This allows us to view surfaces $F \subset B^4$ by their level sets $F_i = F \cap (S^3 \times \{i\})$.

Example: A torus in the 4-ball might look like:



Takeaway: We can answer this question by describing the level sets of a disk D.

Definition of a slice disk

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Definition of a slice disk

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Answer:

Definition of a slice disk

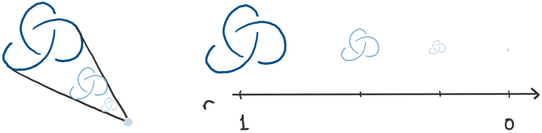
Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Answer: Yes, always!

Definition of a slice disk

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

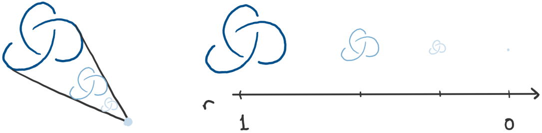
Answer: Yes, always!



Definition of a slice disk

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Answer: Yes, always!

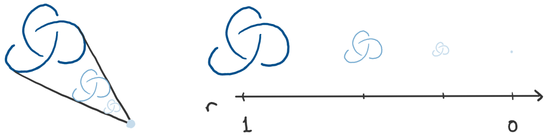


Classic Question:

Definition of a slice disk

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Answer: Yes, always!

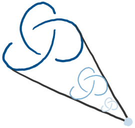


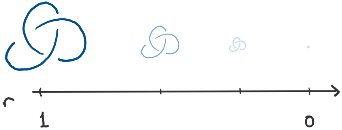
Classic Question:
Given a knot K in the 3-sphere S^3, when does K bound a **smooth** disk D properly embedded in the 4-ball B^4?

Definition of a slice disk

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Answer: Yes, always!





Classic Question:
Given a knot K in the 3-sphere S^3, when does K bound a **smooth** disk D properly embedded in the 4-ball B^4?

Definition

A knot $K \subset S^3$ that bounds a smooth, properly embedded disk $D \subset B^4$ is a **slice knot** and D is a **slice disk**.

Example of a slice disk

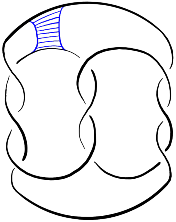
Example:

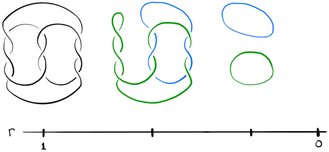
Example of a slice disk

Example:
The knot 9_{46} is slice, with slice disk D_ℓ described by the following level sets:

Example of a slice disk

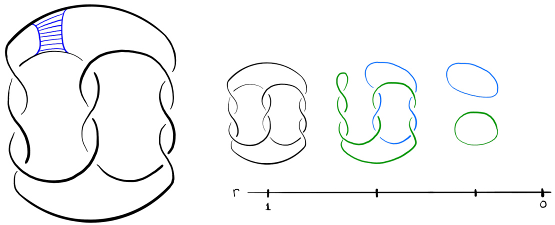
Example:
The knot 9_{46} is slice, with slice disk D_ℓ described by the following level sets:





Example of a slice disk

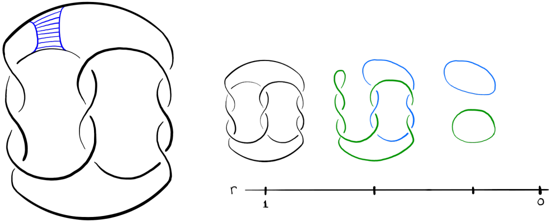
Example:
The knot 9_{46} is slice, with slice disk D_ℓ described by the following level sets:



A second slice D_r can be described similarly, by performing the *band move* on the right-hand-side of 9_{46}.

Example of a slice disk

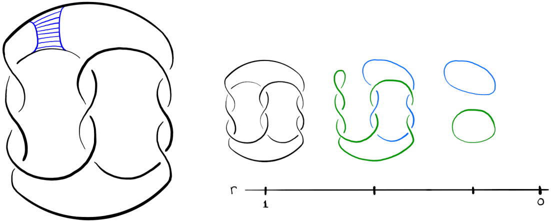
Example:
The knot 9_{46} is slice, with slice disk D_ℓ described by the following level sets:



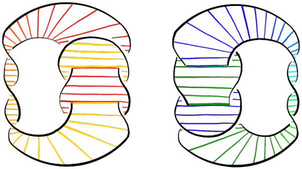
A second slice D_r can be described similarly, by performing the *band move* on the right-hand-side of 9_{46}. We can see these disks pushed into S^3 as:

Example of a slice disk

Example:
The knot 9_{46} is slice, with slice disk D_ℓ described by the following level sets:



A second slice D_r can be described similarly, by performing the *band move* on the right-hand-side of 9_{46}. We can see these disks pushed into S^3 as:



Equivalence of slice disks

Follow-up Question:

Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

Equivalence of slice disks

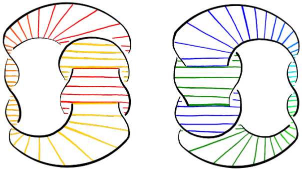
Follow-up Question:
Are D_ℓ and D_r isotopic?

Answer:

Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

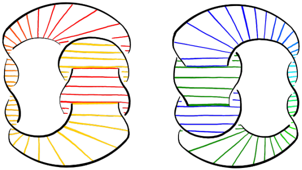
Answer:
Yes - by a rotation!



Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

Answer:
Yes - by a rotation!

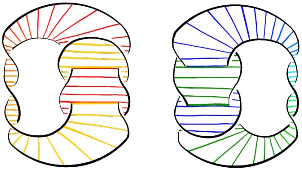


Follow-up Question:

Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

Answer:
Yes - by a rotation!

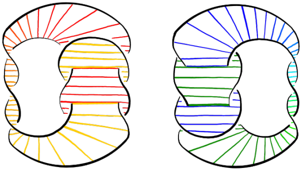


Follow-up Question:
Are D_ℓ and D_r isotopic rel boundary (i.e. leaving 9_{46} fixed)?

Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

Answer:
Yes - by a rotation!



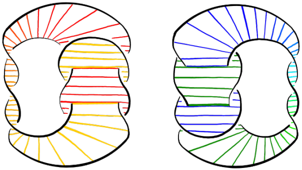
Follow-up Question:
Are D_ℓ and D_r isotopic rel boundary (i.e. leaving 9_{46} fixed)?

Answer:

Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

Answer:
Yes - by a rotation!



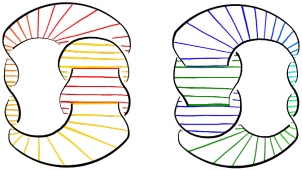
Follow-up Question:
Are D_ℓ and D_r isotopic rel boundary (i.e. leaving 9_{46} fixed)?

Answer:
Maybe? Not exactly easy to tell without doing some math...

Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

Answer:
Yes - by a rotation!



Follow-up Question:
Are D_ℓ and D_r isotopic rel boundary (i.e. leaving 9_{46} fixed)?

Answer:
Maybe? Not exactly easy to tell without doing some math...

We need techniques for studying surfaces up to boundary-preserving isotopy!

Methods for studying slice disks

There are multiple ways to study slice disks up to boundary-preserving isotopy:

Methods for studying slice disks

There are multiple ways to study slice disks up to boundary-preserving isotopy:

- fundamental group of the compliment (e.g. Auckly-Kim-Melvin-Ruberman)

Methods for studying slice disks

There are multiple ways to study slice disks up to boundary-preserving isotopy:

- fundamental group of the compliment (e.g. Auckly-Kim-Melvin-Ruberman)
- Alexander modules (e.g. Miller-Powell)

Methods for studying slice disks

There are multiple ways to study slice disks up to boundary-preserving isotopy:

- fundamental group of the compliment (e.g. Auckly-Kim-Melvin-Ruberman)
- Alexander modules (e.g. Miller-Powell)
- gauge theory (e.g. Akbulut)

Methods for studying slice disks

There are multiple ways to study slice disks up to boundary-preserving isotopy:

- fundamental group of the compliment (e.g. Auckly-Kim-Melvin-Ruberman)
- Alexander modules (e.g. Miller-Powell)
- gauge theory (e.g. Akbulut)
- knot Floer homology (e.g. Juhasz-Zemke)

Methods for studying slice disks

There are multiple ways to study slice disks up to boundary-preserving isotopy:

- fundamental group of the compliment (e.g. Auckly-Kim-Melvin-Ruberman)
- Alexander modules (e.g. Miller-Powell)
- gauge theory (e.g. Akbulut)
- knot Floer homology (e.g. Juhasz-Zemke)
- Khovanov homology

Table of Contents

1 Motivation

2 Khovanov homology of surfaces

3 Khovanov homology of knotted surfaces

4 Khovanov homology of slice disks: Khovanov-Jacobsson classes

5 Khovanov homology of slice disks: reverse cobordisms

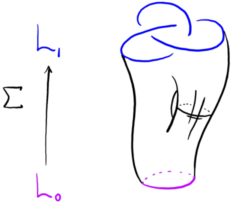
6 Future work

Link cobordisms

Definition. A **link cobordism** $\Sigma\colon L_0 \to L_1$ is a smooth, compact, oriented, properly embedded surface $\Sigma \subset S^3 \times [0,1]$ with boundary a pair ($i \in \{0,1\}$) of oriented links $L_i = \Sigma \cap (S^3 \times \{i\})$.

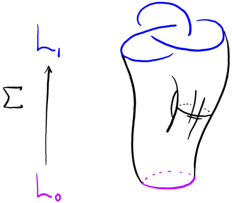
Link cobordisms

Definition. A **link cobordism** $\Sigma\colon L_0 \to L_1$ is a smooth, compact, oriented, properly embedded surface $\Sigma \subset S^3 \times [0,1]$ with boundary a pair ($i \in \{0,1\}$) of oriented links $L_i = \Sigma \cap (S^3 \times \{i\})$.



Link cobordisms

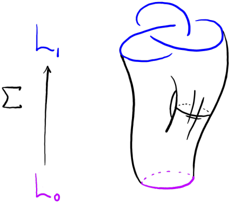
Definition. A **link cobordism** $\Sigma\colon L_0 \to L_1$ is a smooth, compact, oriented, properly embedded surface $\Sigma \subset S^3 \times [0,1]$ with boundary a pair ($i \in \{0,1\}$) of oriented links $L_i = \Sigma \cap (S^3 \times \{i\})$.



Examples: slices ($\emptyset \to K$), closed surfaces ($\emptyset \to \emptyset$), Seifert surfaces ($\emptyset \to K$)

Link cobordisms

Definition. A **link cobordism** $\Sigma\colon L_0 \to L_1$ is a smooth, compact, oriented, properly embedded surface $\Sigma \subset S^3 \times [0,1]$ with boundary a pair ($i \in \{0,1\}$) of oriented links $L_i = \Sigma \cap (S^3 \times \{i\})$.



Examples: slices ($\emptyset \to K$), closed surfaces ($\emptyset \to \emptyset$), Seifert surfaces ($\emptyset \to K$)

Definition. A link cobordism $\Sigma\colon L_0 \to L_1$ can be represented as a **movie**: a finite sequence of diagrams $\{D_{t_i}\}_{i=0}^{n}$, with each successive pair related by an isotopy, Morse move, or Reidemeister move.

Idea of Khovanov homology

Khovanov homology is a *functor* on link cobordisms.

Idea of Khovanov homology

Khovanov homology is a *functor* on link cobordisms.

- links are assigned chain complexes with associated homology groups (or more generally, R-modules)

Idea of Khovanov homology

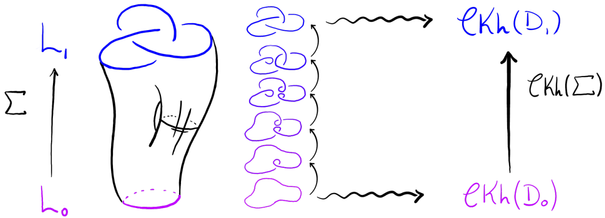
Khovanov homology is a *functor* on link cobordisms.

- links are assigned chain complexes with associated homology groups (or more generally, R-modules)
- link cobordisms are assigned chain maps with induced homomorphisms (or more generally, R-linear maps)

Idea of Khovanov homology

Khovanov homology is a *functor* on link cobordisms.

- links are assigned chain complexes with associated homology groups (or more generally, R-modules)
- link cobordisms are assigned chain maps with induced homomorphisms (or more generally, R-linear maps)



Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathsf{Kh}(D)$ with homology $\mathsf{Kh}(D)$, called the Khovanov homology.

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathsf{Kh}(D)$ with homology $\mathsf{Kh}(D)$, called the Khovanov homology.

How do we define this chain complex?

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathrm{Kh}(D)$ with homology $\mathrm{Kh}(D)$, called the Khovanov homology.

How do we define this chain complex?

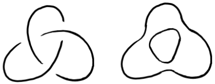
- Choose a diagram D for your link

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathrm{Kh}(D)$ with homology $\mathrm{Kh}(D)$, called the Khovanov homology.

How do we define this chain complex?



- Choose a diagram D for your link
- Smooth each crossing $\times$ in D as a 0-smoothing $\asymp$ or a 1-smoothing)(

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathrm{Kh}(D)$ with homology $\mathrm{Kh}(D)$, called the Khovanov homology.

How do we define this chain complex?

- Choose a diagram D for your link
- Smooth each crossing ╳ in D as a 0-smoothing ≍ or a 1-smoothing)(
- Label each resulting component with a 1 or an x

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathrm{Kh}(D)$ with homology $\mathrm{Kh}(D)$, called the Khovanov homology.

How do we define this chain complex?

- Choose a diagram D for your link
- Smooth each crossing $\times$ in D as a 0-smoothing $\asymp$ or a 1-smoothing $)($
- Label each resulting component with a 1 or an x
- Generate $\mathcal{C}\mathrm{Kh}(D)$ over $\mathbb{Z}$ with all possible *labeled smoothings*

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathsf{Kh}(D)$ with homology $\mathsf{Kh}(D)$, called the Khovanov homology.

Properties:

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathrm{Kh}(D)$ with homology $\mathrm{Kh}(D)$, called the Khovanov homology.

Properties:

- Different diagrams have isomorphic Khovanov homology
  (we write $\mathrm{Kh}(L)$ to mean: choose a diagram D for L and consider $\mathrm{Kh}(D)$)

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathsf{Kh}(D)$ with homology $\mathsf{Kh}(D)$, called the Khovanov homology.

Properties:

- Different diagrams have isomorphic Khovanov homology (we write $\mathsf{Kh}(L)$ to mean: choose a diagram D for L and consider $\mathsf{Kh}(D)$)
- We set $\mathcal{C}\mathsf{Kh}(\emptyset) = \mathbb{Z}$ and $\mathsf{Kh}(\emptyset) = \mathbb{Z}$

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathsf{Kh}(D)$ with homology $\mathsf{Kh}(D)$, called the Khovanov homology.

Properties:

- Different diagrams have isomorphic Khovanov homology (we write $\mathsf{Kh}(L)$ to mean: choose a diagram D for L and consider $\mathsf{Kh}(D)$)
- We set $\mathcal{C}\mathsf{Kh}(\emptyset) = \mathbb{Z}$ and $\mathsf{Kh}(\emptyset) = \mathbb{Z}$
- There is a bigrading $\mathcal{C}\mathsf{Kh}^{h,q}(D)$

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathrm{Kh}(D)$ with homology $\mathrm{Kh}(D)$, called the Khovanov homology.

Properties:

- Different diagrams have isomorphic Khovanov homology (we write $\mathrm{Kh}(L)$ to mean: choose a diagram D for L and consider $\mathrm{Kh}(D)$)
- We set $\mathcal{C}\mathrm{Kh}(\emptyset) = \mathbb{Z}$ and $\mathrm{Kh}(\emptyset) = \mathbb{Z}$
- There is a bigrading $\mathcal{C}\mathrm{Kh}^{h,q}(D)$
- There is a differential $d\colon \mathcal{C}\mathrm{Kh}^{h,q}(D) \to \mathcal{C}\mathrm{Kh}^{h+1,q}(D)$

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

How do we define these chain maps?

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

How do we define these chain maps?

- The diagrams D_{t_i} in the movie each have an associated chain complex

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

How do we define these chain maps?

- The diagrams D_{t_i} in the movie each have an associated chain complex
- Adjacent frames $D_{t_i} \to D_{t_{i+1}}$ are related by an isotopy, Morse move, or Reidemeister moves

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^n$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

How do we define these chain maps?

- The diagrams D_{t_i} in the movie each have an associated chain complex
- Adjacent frames $D_{t_i} \to D_{t_{i+1}}$ are related by an isotopy, Morse move, or Reidemeister moves
- Define chain maps for each of these moves

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

How do we define these chain maps?

- The diagrams D_{t_i} in the movie each have an associated chain complex
- Adjacent frames $D_{t_i} \to D_{t_{i+1}}$ are related by an isotopy, Morse move, or Reidemeister moves
- Define chain maps for each of these moves
- Compose these chain maps to produce $\mathcal{C}\mathsf{Kh}(\Sigma)$

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

Properties:

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

Properties:

- This map is also bigraded:

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}^{h,q}(D_0) \to \mathcal{C}\mathsf{Kh}^{h,q+\chi(\Sigma)}(D_1)$$

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^n$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

Properties:

- This map is also bigraded:

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}^{h,q}(D_0) \to \mathcal{C}\mathsf{Kh}^{h,q+\chi(\Sigma)}(D_1)$$

- Generally, they are difficult to compute...

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

Properties:

- This map is also bigraded:

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}^{h,q}(D_0) \to \mathcal{C}\mathsf{Kh}^{h,q+\chi(\Sigma)}(D_1)$$

- Generally, they are difficult to compute...
- But they have one very useful property!

Invariance

Theorem (Jacobsson '04, Bar-Natan '05, Khovanov '06)

The map on Khovanov homology induced by a link cobordism Σ is invariant, up to sign, under smooth boundary-preserving isotopy of Σ.

Invariance

Theorem (Jacobsson '04, Bar-Natan '05, Khovanov '06)

The map on Khovanov homology induced by a link cobordism Σ is invariant, up to sign, under smooth boundary-preserving isotopy of Σ.

Goal:

Invariance

Theorem (Jacobsson '04, Bar-Natan '05, Khovanov '06)

The map on Khovanov homology induced by a link cobordism Σ is invariant, up to sign, under smooth boundary-preserving isotopy of Σ.

Goal:
Distinguish link cobordisms Σ, Σ' up to **smooth** isotopy rel boundary by showing their induced maps are distinct $\mathrm{Kh}(\Sigma) \neq \pm\mathrm{Kh}(\Sigma')$

Invariance

Theorem (Jacobsson '04, Bar-Natan '05, Khovanov '06)

The map on Khovanov homology induced by a link cobordism Σ is invariant, up to sign, under smooth boundary-preserving isotopy of Σ.

Goal:
Distinguish link cobordisms Σ, Σ' up to **smooth** isotopy rel boundary by showing their induced maps are distinct $\mathsf{Kh}(\Sigma) \neq \pm\mathsf{Kh}(\Sigma')$

$$\mathsf{Kh}(\Sigma) \neq \pm\mathsf{Kh}(\Sigma') \implies \Sigma \not\simeq_\partial \Sigma'$$

Table of Contents

1. Motivation
2. Khovanov homology of surfaces
3. Khovanov homology of knotted surfaces
4. Khovanov homology of slice disks: Khovanov-Jacobsson classes
5. Khovanov homology of slice disks: reverse cobordisms
6. Future work

Khovanov-Jacobsson numbers

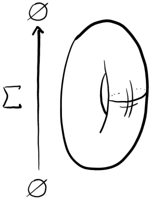
Question:

Khovanov-Jacobsson numbers

Question:
Can these induced maps distinguish (closed) knotted surfaces in B^4?

Khovanov-Jacobsson numbers

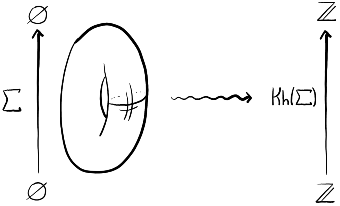
Question:
Can these induced maps distinguish (closed) knotted surfaces in B^4?



- A knotted surface Σ can be regarded as a link cobordism $\Sigma\colon \emptyset \to \emptyset$.

Khovanov-Jacobsson numbers

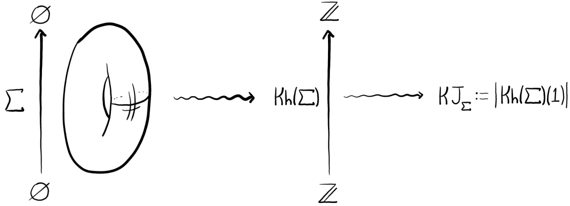
Question:
Can these induced maps distinguish (closed) knotted surfaces in B^4?



- A knotted surface Σ can be regarded as a link cobordism $\Sigma\colon \emptyset \to \emptyset$.
- It induces a map Kh$(\Sigma)\colon \mathbb{Z} \to \mathbb{Z}$

Khovanov-Jacobsson numbers

Question:
Can these induced maps distinguish (closed) knotted surfaces in B^4?



- A knotted surface Σ can be regarded as a link cobordism $\Sigma\colon \emptyset \to \emptyset$.
- It induces a map $\mathrm{Kh}(\Sigma)\colon \mathbb{Z} \to \mathbb{Z}$
- This map is determined by $\mathrm{Kh}(\Sigma)(1) \in \mathbb{Z}$, so this integer is an up-to-sign invariant of the (ambient) isotopy class of Σ

Rasmussen-Tanaka

Lemma

For a link cobordism $\Sigma\colon \emptyset \to \emptyset$, the Khovanov-Jacobsson number

$$\mathsf{KJ}_\Sigma := |\mathsf{Kh}(\Sigma)(1)| \in \mathbb{Z}$$

is an invariant of the ambient isotopy class of Σ.

Rasmussen-Tanaka

Lemma

For a link cobordism $\Sigma\colon \emptyset \to \emptyset$, the Khovanov-Jacobsson number

$$\mathsf{KJ}_\Sigma := |\mathsf{Kh}(\Sigma)(1)| \in \mathbb{Z}$$

is an invariant of the ambient isotopy class of Σ.

Question. Do Khovanov-Jacobsson numbers distinguish any knotted surfaces?

Rasmussen-Tanaka

Lemma

For a link cobordism $\Sigma\colon \emptyset \to \emptyset$, the Khovanov-Jacobsson number

$$\mathsf{KJ}_\Sigma := |\mathsf{Kh}(\Sigma)(1)| \in \mathbb{Z}$$

is an invariant of the ambient isotopy class of Σ.

Question. Do Khovanov-Jacobsson numbers distinguish any knotted surfaces?

Theorem (Rasmussen '05, Tanaka '05)

Khovanov-Jacobsson numbers of connected Σ are determined by genus:

- *if $g(\Sigma) = 1$, then $\mathsf{KJ}_\Sigma = 2$*
- *if $g(\Sigma) \neq 1$, then $\mathsf{KJ}_\Sigma = 0$*

Cases

Idea:

Cases

Idea:
Follow the same procedure for surfaces with boundary.

Cases

Idea:
Follow the same procedure for surfaces with boundary.

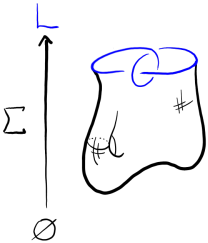
A (nice) surface $\Sigma \subset B^4$ with boundary $L \subset S^3$ can be regarded as:

Cases

Idea:
Follow the same procedure for surfaces with boundary.

A (nice) surface $\Sigma \subset B^4$ with boundary $L \subset S^3$ can be regarded as:

a. a link cobordism $\Sigma \colon \emptyset \to L$, or



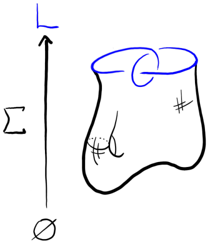
Cases

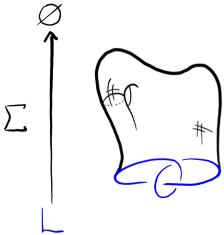
Idea:
Follow the same procedure for surfaces with boundary.

A (nice) surface $\Sigma \subset B^4$ with boundary $L \subset S^3$ can be regarded as:

a. a link cobordism $\Sigma\colon \emptyset \to L$, or

b. its reverse cobordism $\Sigma\colon L \to \emptyset$





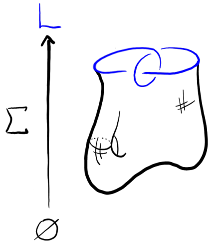
Cases

Idea:
Follow the same procedure for surfaces with boundary.

A (nice) surface $\Sigma \subset B^4$ with boundary $L \subset S^3$ can be regarded as:

a. a link cobordism $\Sigma\colon \emptyset \to L$, or
b. its reverse cobordism $\Sigma\colon L \to \emptyset$

We consider these cases separately.



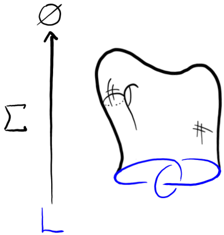


Table of Contents

1 Motivation

2 Khovanov homology of surfaces

3 Khovanov homology of knotted surfaces

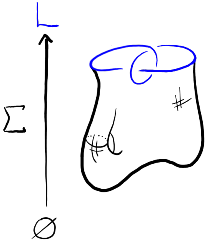
4 Khovanov homology of slice disks: Khovanov-Jacobsson classes

5 Khovanov homology of slice disks: reverse cobordisms

6 Future work

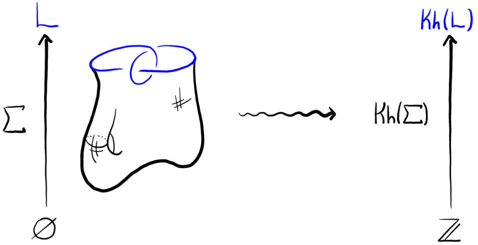
Khovanov-Jacobsson classes

Case 1: Consider a link cobordism $\Sigma \colon \emptyset \to L$



Khovanov-Jacobsson classes

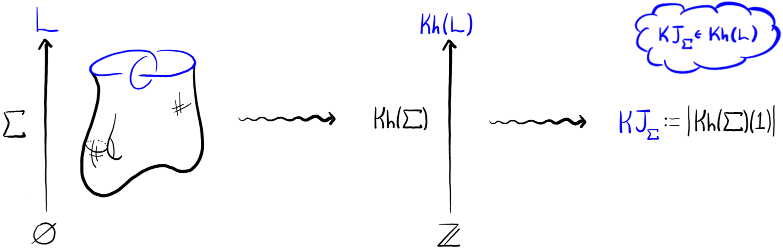
Case 1: Consider a link cobordism $\Sigma\colon \emptyset \to L$



Consider the induced map $\mathsf{Kh}(\Sigma)\colon \mathbb{Z} \to \mathsf{Kh}(L)$

Khovanov-Jacobsson classes

Case 1: Consider a link cobordism $\Sigma\colon \emptyset \to L$



Consider the induced map $\mathsf{Kh}(\Sigma)\colon \mathbb{Z} \to \mathsf{Kh}(L)$

This map is determined by $\mathsf{Kh}(\Sigma)(1) \in \mathsf{Kh}(L)$, so this homology class is an up-to-sign invariant of the (relative) isotopy class of Σ.

Khovanov-Jacobsson classes

Case 1: Consider a link cobordism $\Sigma\colon \emptyset \to L$

Lemma

For a link cobordism $\Sigma\colon \emptyset \to L$, the Khovanov-Jacobsson class

$$\mathsf{KJ}_\Sigma := |\mathsf{Kh}(\Sigma)(1)| \in \mathsf{Kh}(L)$$

is an invariant of the boundary-preserving isotopy class of Σ.

Khovanov-Jacobsson classes

Case 1: Consider a link cobordism $\Sigma \colon \emptyset \to L$

Lemma

For a link cobordism $\Sigma \colon \emptyset \to L$, the Khovanov-Jacobsson class

$$\mathsf{KJ}_\Sigma := |\mathsf{Kh}(\Sigma)(1)| \in \mathsf{Kh}(L)$$

is an invariant of the boundary-preserving isotopy class of Σ.

Question:
Do Khovanov-Jacobsson classes distinguish any surfaces?

Khovanov-Jacobsson classes

Case 1: Consider a link cobordism $\Sigma\colon \emptyset \to L$

Lemma

For a link cobordism $\Sigma\colon \emptyset \to L$, the Khovanov-Jacobsson class

$$\mathsf{KJ}_\Sigma := |\mathsf{Kh}(\Sigma)(1)| \in \mathsf{Kh}(L)$$

is an invariant of the boundary-preserving isotopy class of Σ.

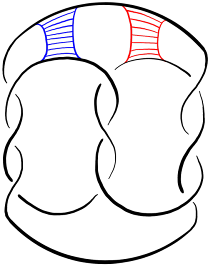
Question:
Do Khovanov-Jacobsson classes distinguish any surfaces?

Hopefully!

Khovanov-Jacobsson classes

Theorem (Swann '10, S. '20)

The slice disks D_ℓ and D_r for 9_{46} have distinct Khovanov-Jacobsson classes $\mathrm{KJ}_{D_\ell} \neq \mathrm{KJ}_{D_r}$, and therefore, are not isotopic rel boundary.



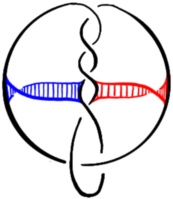
Khovanov-Jacobsson classes

Theorem (Swann '10, S. '20)

The slice disks D_ℓ and D_r for 9_{46} have distinct Khovanov-Jacobsson classes $\mathsf{KJ}_{D_\ell} \neq \mathsf{KJ}_{D_r}$, and therefore, are not isotopic rel boundary.

Theorem (S. '20)

The slice disks D_ℓ and D_r for 6_1 (below) have distinct Khovanov-Jacobsson classes $\mathsf{KJ}_{D_\ell} \neq \mathsf{KJ}_{D_r}$, and therefore, are not isotopic rel boundary.



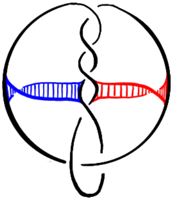
Khovanov-Jacobsson classes

Theorem (Swann '10, S. '20)

The slice disks D_ℓ and D_r for 9_{46} have distinct Khovanov-Jacobsson classes $\mathsf{KJ}_{D_\ell} \neq \mathsf{KJ}_{D_r}$*, and therefore, are not isotopic rel boundary.*

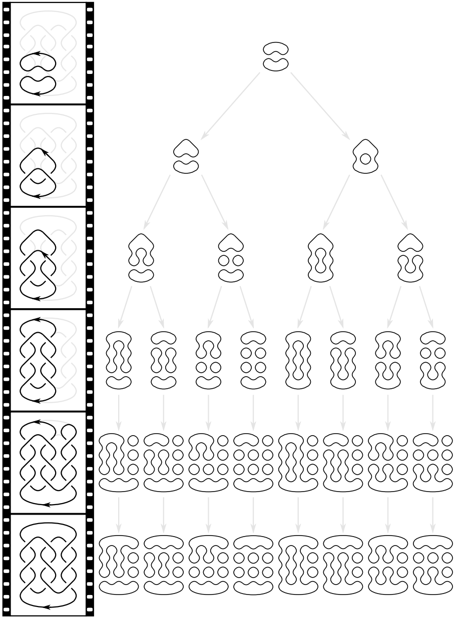
Theorem (S. '20)

The slice disks D_ℓ and D_r for 6_1 (below) have distinct Khovanov-Jacobsson classes $\mathsf{KJ}_{D_\ell} \neq \mathsf{KJ}_{D_r}$*, and therefore, are not isotopic rel boundary.*



Note: this uniqueness is also known through other techniques.

Calculation for 9_{46}



Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

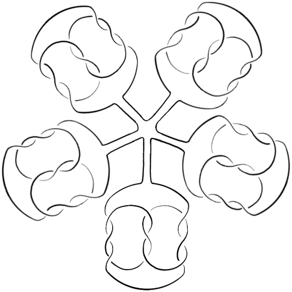
The 2^n *slices of* $\#_n(9_{46})$ *have distinct Khovanov-Jacobsson classes, and therefore, are not isotopic rel boundary.*

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

The 2^n slices of $\#_n(9_{46})$ have distinct Khovanov-Jacobsson classes, and therefore, are not isotopic rel boundary.

Slices are obtained by choosing one of the band moves for each copy of 9_{46} (or boundary connect summing the slices).

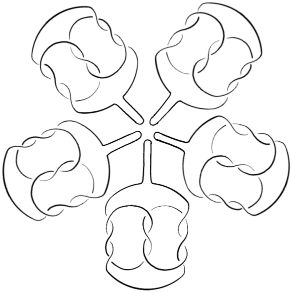


Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

The 2^n slices of $\#_n(9_{46})$ have distinct Khovanov-Jacobsson classes, and therefore, are not isotopic rel boundary.

Slices are obtained by choosing one of the band moves for each copy of 9_{46} (or boundary connect summing the slices).

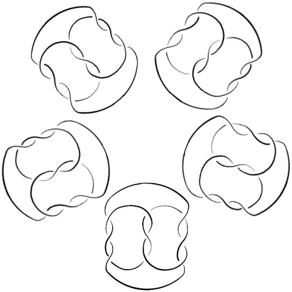


Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

The 2^n slices of $\#_n(9_{46})$ have distinct Khovanov-Jacobsson classes, and therefore, are not isotopic rel boundary.

Slices are obtained by choosing one of the band moves for each copy of 9_{46} (or boundary connect summing the slices).

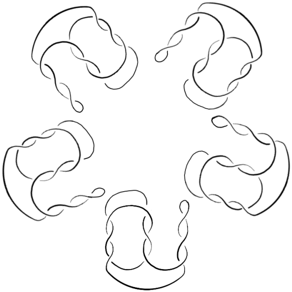


Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

The 2^n slices of $\#_n(9_{46})$ have distinct Khovanov-Jacobsson classes, and therefore, are not isotopic rel boundary.

Slices are obtained by choosing one of the band moves for each copy of 9_{46} (or boundary connect summing the slices).

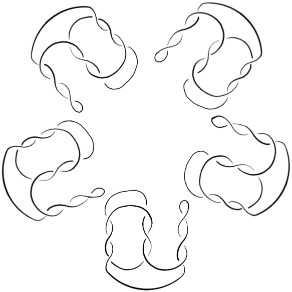


Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

The 2^n slices of $\#_n(9_{46})$ have distinct Khovanov-Jacobsson classes, and therefore, are not isotopic rel boundary.

Slices are obtained by choosing one of the band moves for each copy of 9_{46} (or boundary connect summing the slices).



This can also be done with $\#_n(6_1)$, or even by using combinations of 9_{46} and 6_1.

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

- Every knot is ribbon concordant to a prime knot (Kirby-Lickorish)

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

- Every knot is ribbon concordant to a prime knot (Kirby-Lickorish)
- Ribbon concordances induce injections on Khovanov homology (Levine-Zemke)

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

- Every knot is ribbon concordant to a prime knot (Kirby-Lickorish)
- Ribbon concordances induce injections on Khovanov homology (Levine-Zemke)
- So, extend the 2^n slices for $\#_n(9_{46})$ by a ribbon-concordance to a prime knot

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

- Every knot is ribbon concordant to a prime knot (Kirby-Lickorish)
- Ribbon concordances induce injections on Khovanov homology (Levine-Zemke)
- So, extend the 2^n slices for $\#_n(9_{46})$ by a ribbon-concordance to a prime knot
- Slices will continue to have distinct Khovanov-Jacobsson classes

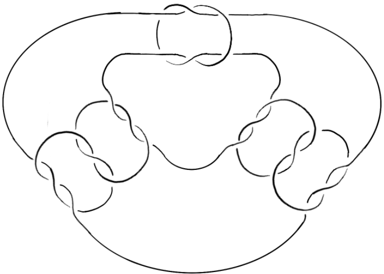
Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

- Every knot is ribbon concordant to a prime knot (Kirby-Lickorish)
- Ribbon concordances induce injections on Khovanov homology (Levine-Zemke)
- So, extend the 2^n slices for $\#_n(9_{46})$ by a ribbon-concordance to a prime knot
- Slices will continue to have distinct Khovanov-Jacobsson classes



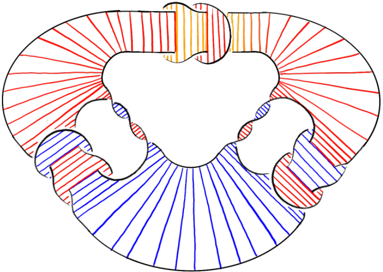
Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

- Every knot is ribbon concordant to a prime knot (Kirby-Lickorish)
- Ribbon concordances induce injections on Khovanov homology (Levine-Zemke)
- So, extend the 2^n slices for $\#_n(9_{46})$ by a ribbon-concordance to a prime knot
- Slices will continue to have distinct Khovanov-Jacobsson classes



Application: Obstructing sliceness

Application: Obstructing sliceness

Theorem (Swann '10)

If $\Sigma\colon \emptyset \to K$ *has genus* $g(\Sigma) = 1$ *and* $\mathsf{KJ}_\Sigma = 0$ *then* K *is not slice.*

Application: Obstructing sliceness

Theorem (Swann '10)

If $\Sigma\colon \emptyset \to K$ *has genus* $g(\Sigma) = 1$ *and* $\mathsf{KJ}_\Sigma = 0$ *then* K *is not slice.*

Proof idea: assume K has a slice disk D and apply the absolute case to $D \circ \Sigma$.

Application: Obstructing sliceness

Theorem (Swann '10)

If $\Sigma\colon \emptyset \to K$ has genus $g(\Sigma) = 1$ and $\mathsf{KJ}_\Sigma = 0$ then K is not slice.

Proof idea: assume K has a slice disk D and apply the absolute case to $D \circ \Sigma$.

Note: there are classes of knots with 4-ball genus at most 1 (e.g. Whitehead doubles, unknotting number 1 knots)

Application: Obstructing sliceness

Theorem (Swann '10)

If $\Sigma\colon \emptyset \to K$ *has genus* $g(\Sigma) = 1$ *and* $\mathrm{KJ}_\Sigma = 0$ *then* K *is not slice.*

Proof idea: assume K has a slice disk D and apply the absolute case to $D \circ \Sigma$.

Note: there are classes of knots with 4-ball genus at most 1 (e.g. Whitehead doubles, unknotting number 1 knots)

Corollary (Swann '10)

For $p, q, r \geq 3$ *and odd, the pretzel knot* $P(p, q, r)$ *is not slice.*

Application: Obstructing sliceness

Theorem (Swann '10)

If $\Sigma\colon \emptyset \to K$ has genus $g(\Sigma) = 1$ and $\mathrm{KJ}_\Sigma = 0$ then K is not slice.

Proof idea: assume K has a slice disk D and apply the absolute case to $D \circ \Sigma$.

Note: there are classes of knots with 4-ball genus at most 1 (e.g. Whitehead doubles, unknotting number 1 knots)

Corollary (Swann '10)

For $p, q, r \geq 3$ and odd, the pretzel knot $P(p, q, r)$ is not slice.

Corollary (Swann '10)

For $p, q \leq -3$ and odd, the pretzel knot $P(p, q, 1)$ is not slice.

Are we still on case 1?

Downside to Khovanov-Jacobsson classes:

Are we still on case 1?

Downside to Khovanov-Jacobsson classes:

- Hard to calculate...

Are we still on case 1?

Downside to Khovanov-Jacobsson classes:

- Hard to calculate...
- Hard to distinguish...

Are we still on case 1?

Downside to Khovanov-Jacobsson classes:

- Hard to calculate...
- Hard to distinguish...

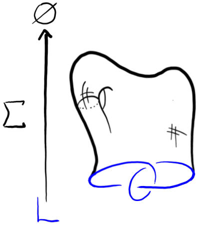
Is there a better way?

Table of Contents

1. Motivation
2. Khovanov homology of surfaces
3. Khovanov homology of knotted surfaces
4. Khovanov homology of slice disks: Khovanov-Jacobsson classes
5. Khovanov homology of slice disks: reverse cobordisms
6. Future work

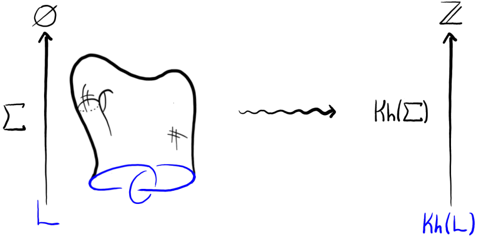
Reverse cobordism

Case 2: Consider a link cobordism $\Sigma\colon L \to \emptyset$



Reverse cobordism

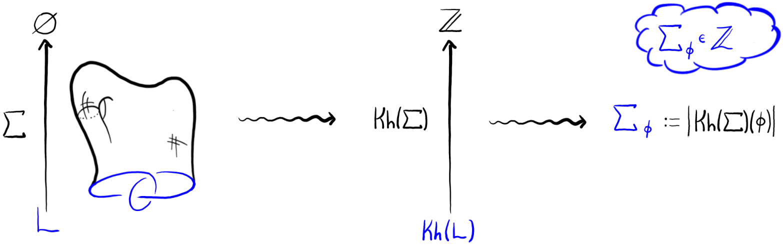
Case 2: Consider a link cobordism $\Sigma\colon L \to \emptyset$



Consider the induced map $\mathsf{Kh}(\Sigma)\colon \mathsf{Kh}(L) \to \mathbb{Z}$

Reverse cobordism

Case 2: Consider a link cobordism $\Sigma\colon L \to \emptyset$



Consider the induced map $\mathsf{Kh}(\Sigma)\colon \mathsf{Kh}(L) \to \mathbb{Z}$

Choose a class $\phi \in \mathsf{Kh}(L)$, and note that $\mathsf{Kh}(\Sigma)(\phi) \in \mathbb{Z}$ is an up-to-sign invariant of the (relative) isotopy class of Σ.

Reverse cobordism

Case 2: Consider a link cobordism $\Sigma\colon L \to \emptyset$

Lemma

For a link cobordism $\Sigma\colon L \to \emptyset$ *and a class* $\phi \in \mathsf{Kh}(L)$*, the integer*

$$\Sigma_\phi := |\mathsf{Kh}(\Sigma)(\phi)| \in \mathbb{Z}$$

is an invariant of the boundary-preserving isotopy class of Σ.

Reverse cobordism

Case 2: Consider a link cobordism $\Sigma\colon L \to \emptyset$

Lemma

For a link cobordism $\Sigma\colon L \to \emptyset$ *and a class* $\phi \in \mathsf{Kh}(L)$*, the integer*

$$\Sigma_\phi := |\mathsf{Kh}(\Sigma)(\phi)| \in \mathbb{Z}$$

is an invariant of the boundary-preserving isotopy class of Σ*.*

Questions:
Do these invariants distinguish any surfaces?

Reverse cobordism

Case 2: Consider a link cobordism $\Sigma\colon L \to \emptyset$

Lemma

For a link cobordism $\Sigma\colon L \to \emptyset$ and a class $\phi \in \mathsf{Kh}(L)$, the integer

$$\Sigma_\phi := |\mathsf{Kh}(\Sigma)(\phi)| \in \mathbb{Z}$$

is an invariant of the boundary-preserving isotopy class of Σ.

Questions:
Do these invariants distinguish any surfaces?
Are they better than Khovanov-Jacobsson classes?

Quick results

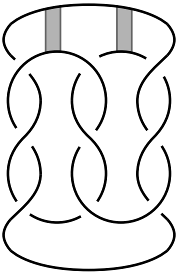
Theorem (Hayden-S. '21)

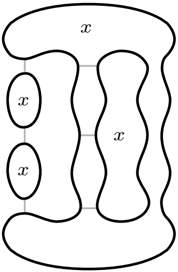
The pair of slice disks D_ℓ and D_r for the knot K (below) induce distinct maps on Khovanov homology, distinguished by the given class $\phi \in \mathsf{Kh}(K)$, and therefore, are not isotopic rel boundary.

Quick results

Theorem (Hayden-S. '21)

The pair of slice disks D_ℓ and D_r for the knot K (below) induce distinct maps on Khovanov homology, distinguished by the given class $\phi \in \mathsf{Kh}(K)$, and therefore, are not isotopic rel boundary.

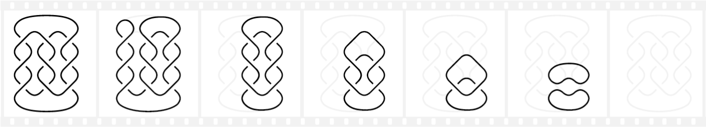


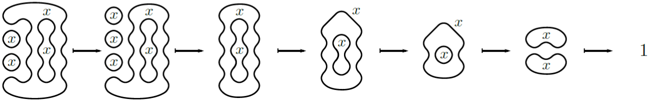


Quick results

Theorem (Hayden-S. '21)

The pair of slice disks D_ℓ and D_r for the knot K (below) induce distinct maps on Khovanov homology, distinguished by the given class $\phi \in \mathrm{Kh}(K)$, and therefore, are not isotopic rel boundary.

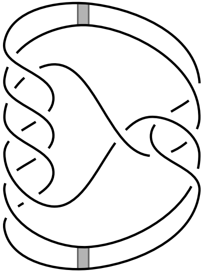




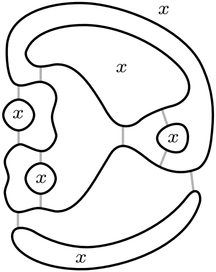
Quick results

Theorem (Hayden-S. '21)

The pair of slice disks D_ℓ and D_r for the knot K (below) induce distinct maps on Khovanov homology, distinguished by the given class $\phi \in \mathsf{Kh}(K)$, and therefore, are not isotopic rel boundary.



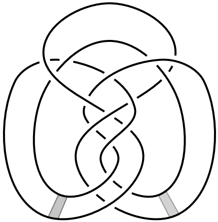
$15n_{103488}$



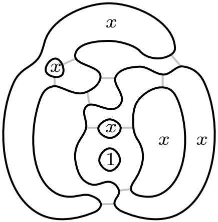
Quick results

Theorem (Hayden-S. '21)

The pair of slice disks D_ℓ and D_r for the knot K (below) induce distinct maps on Khovanov homology, distinguished by the given class $\phi \in \mathsf{Kh}(K)$, and therefore, are not isotopic rel boundary.



$17nh_{74}$



Exotic slices

Fact:
The slices for 6_1, 9_{46}, and $15n_{103488}$ are not even topologically isotopic rel boundary.

Exotic slices

Fact:
The slices for 6_1, 9_{46}, and $15n_{103488}$ are not even topologically isotopic rel boundary.

Definition

A pair of slice disks are *exotic* if they are topologically isotopic rel boundary, but not smoothly isotopic rel boundary.

Exotic slices

Fact:
The slices for 6_1, 9_{46}, and $15n_{103488}$ are not even topologically isotopic rel boundary.

Definition

A pair of slice disks are *exotic* if they are topologically isotopic rel boundary, but not smoothly isotopic rel boundary.

Theorem (Hayden '21)

The slices for $17nh_{74}$ *are topologically isotopic rel boundary.*

Exotic slices

Fact:
The slices for 6_1, 9_{46}, and $15n_{103488}$ are not even topologically isotopic rel boundary.

Definition

A pair of slice disks are *exotic* if they are topologically isotopic rel boundary, but not smoothly isotopic rel boundary.

Theorem (Hayden '21)

The slices for $17nh_{74}$ are topologically isotopic rel boundary.

Corollary (Hayden-S. '21)

The induced maps on Khovanov homology detect exotic pairs of slice disks.

Exotic slices

Fact:
The slices for 6_1, 9_{46}, and $15n_{103488}$ are not even topologically isotopic rel boundary.

Definition

A pair of slice disks are *exotic* if they are topologically isotopic rel boundary, but not smoothly isotopic rel boundary.

Theorem (Hayden '21)

The slices for $17nh_{74}$ *are topologically isotopic rel boundary.*

Corollary (Hayden-S. '21)

The induced maps on Khovanov homology detect exotic pairs of slice disks.

Can be extended to an infinite family of knots bounding pairs of ambiently non-isotopic surfaces of any genus.

Comparisons

Case 1:

- It is hard to compute KJ_Σ
- It is hard to compare KJ_Σ and $\mathrm{KJ}_{\Sigma'}$

Comparisons

Case 1:

- It is hard to compute KJ_{Σ}
- It is hard to compare KJ_{Σ} and $\mathrm{KJ}_{\Sigma'}$

Case 2:

- By choosing ϕ wisely, it is easier to compute Σ_{ϕ}
- Comparing integers is easy

Table of Contents

1 Motivation

2 Khovanov homology of surfaces

3 Khovanov homology of knotted surfaces

4 Khovanov homology of slice disks: Khovanov-Jacobsson classes

5 Khovanov homology of slice disks: reverse cobordisms

6 Future work

Future work

- explore relationship between KJ-classes and reverse cobordisms

Future work

- explore relationship between KJ-classes and reverse cobordisms
- tweak the algebra (e.g. annular Khovanov homology)

Future work

- explore relationship between KJ-classes and reverse cobordisms
- tweak the algebra (e.g. annular Khovanov homology)
- tweak the topology (slice disks in different 4-manifolds)

Future work

- explore relationship between KJ-classes and reverse cobordisms
- tweak the algebra (e.g. annular Khovanov homology)
- tweak the topology (slice disks in different 4-manifolds)
- study different families of disks (rolling, spinning, symmetries)

Future work

- explore relationship between KJ-classes and reverse cobordisms
- tweak the algebra (e.g. annular Khovanov homology)
- tweak the topology (slice disks in different 4-manifolds)
- study different families of disks (rolling, spinning, symmetries)
- study relationship with other invariants (e.g. s-invariant or knot Floer homology)

Future work

- explore relationship between KJ-classes and reverse cobordisms
- tweak the algebra (e.g. annular Khovanov homology)
- tweak the topology (slice disks in different 4-manifolds)
- study different families of disks (rolling, spinning, symmetries)
- study relationship with other invariants (e.g. s-invariant or knot Floer homology)
- study slice obstruction from Khovanov-Jacobsson classes

Thank You!

Thank you!

Bibliography I

- D Bar-Natan, *Khovanov's homology for tangles and cobordisms*, **Geom. Topol.**, 9:1443-1499, 2005.
- *Characterisation of homotopy ribbon discs*, **Adv. Math.**, 391:Paper No. 107960, 2021.
- Kyle Hayden, *Corks, covers, and complex curves*, arXiv:2107.06856, 2021.
- Kyle Hayden and Isaac Sundberg, *Khovanov homology and exotic surfaces in the 4-ball*, arXiv:2108.04810, 2021.
- Magnus Jacobsson, *An invariant of link cobordisms from Khovanov homology*, **Algebr. Geom. Topol.**, 4:1211-1251, 2004.
- András Juhász and Ian Zemke, *Distinguishing slice disks using knot floer homology*, *Seceta Math. (N.S.)*,20(1), 2020.
- Mikhail Khovanov, *A categorification of the Jones polynomial*, **Duke Math. J.**, 101(3):359-426, 2000.
- Mikhail Khovanov, *An invariant of tangle cobordisms*, **Transactions of the American Mathematical Society**, 358(1):315-327, 2006.

Bibliography II

- Adam Simon Levine and Ian Zemke, *Khovanov homology and ribbon concordances*, **Bull. Lond. Math. Soc.**, 51(6):1099-1103, 2019.
- Allison N. Miller and Mark Powell, Stabilization distance between surfaces, *Enseign. Math.*,**65**:397-440, 2020.
- Lisa Piccirillo, *The Conway knot is not slice*, **Ann. of Math.** (2), 191(2):581-591, 2020.
- Jacob Rasmussen, *Khovanov's invariant for closed surfaces*, arXiv:math/0502527, 2005.
- Isaac Sundberg and Jonah Swann, *Relative Khovanov-Jacobsson classes*, arXiv:2103.01438, 2021.
- Jonah Swann, *Relative Khovanov-Jacobsson classes of spanning surfaces*, *Ph.D. Thesis, Bryn Mawr College*, 2010.
- Kokoro Tanaka, *Khovanov-Jacobsson numbers and invariants of surface-knots derived from Bar-Natan's theory*, **Proc. Amer. Math. Soc.**, 134(12):3685–3689, 2005.