The knots so nice they sliced them twice

Detecting exoticity with Khovanov homology

Isaac Sundberg

Bryn Mawr College

Dissertation Defense

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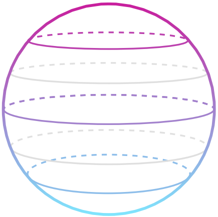

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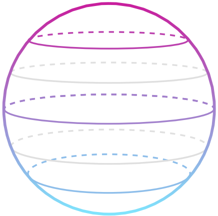

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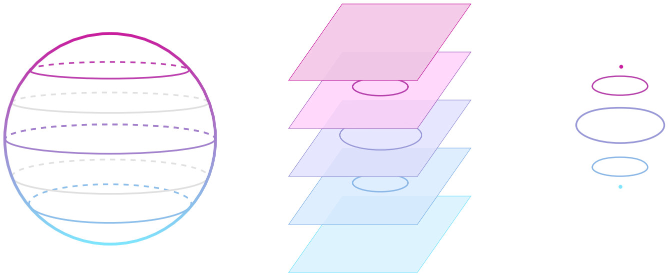

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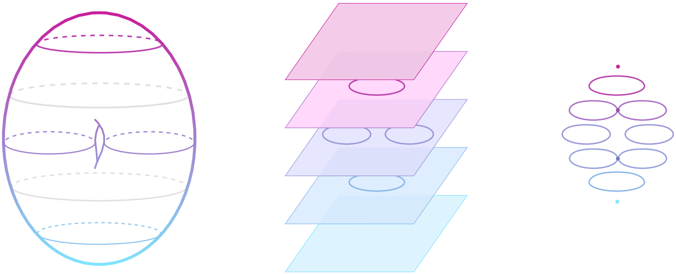

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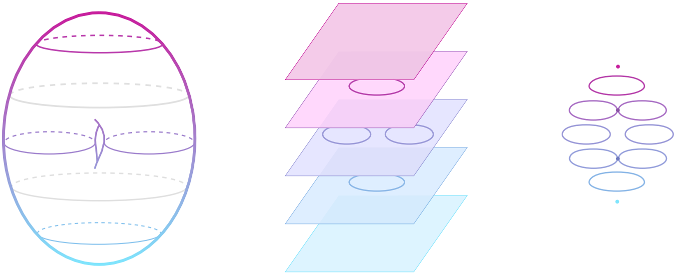

Let's do the same thing for surfaces in $\mathbb{R}^{3+1}$

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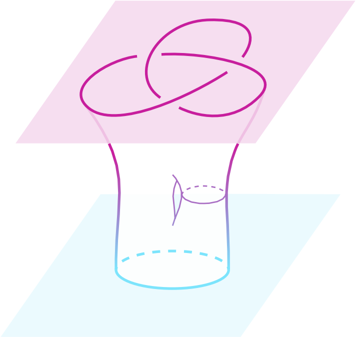

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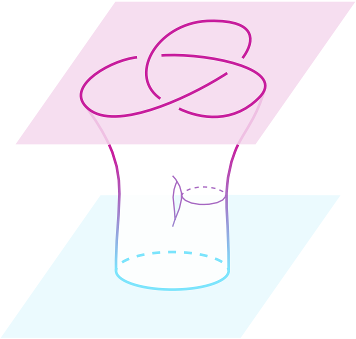

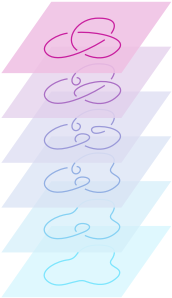

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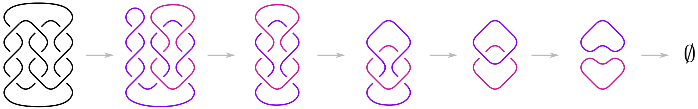

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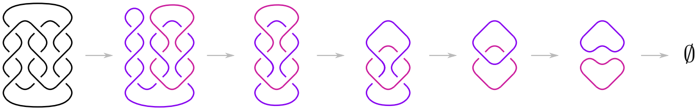

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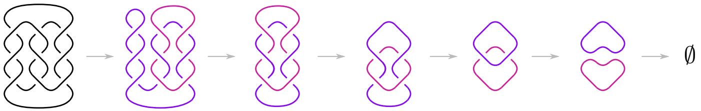

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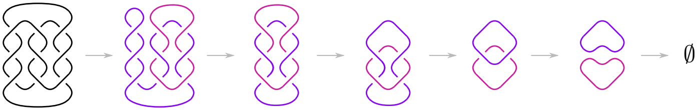

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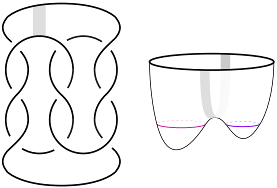

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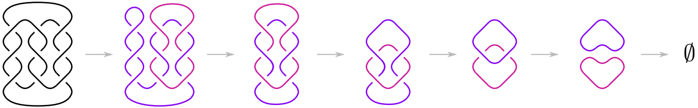

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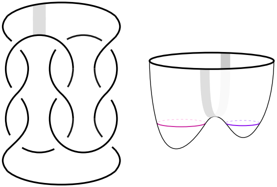

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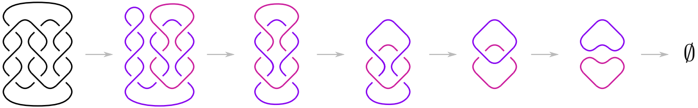

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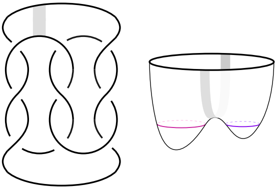

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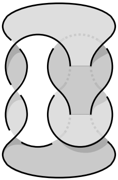

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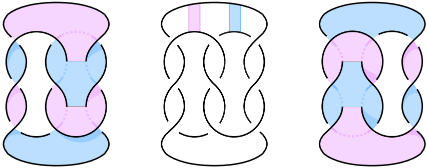

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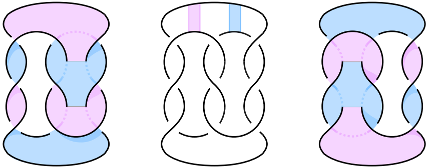

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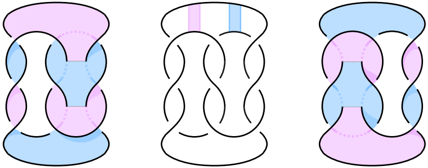

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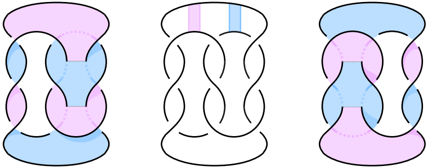

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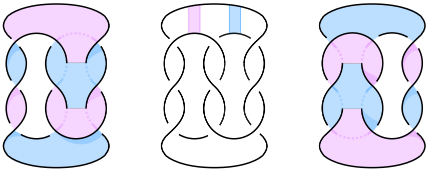

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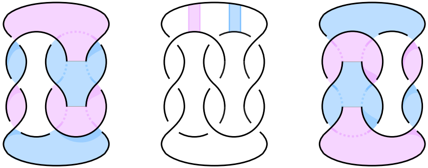

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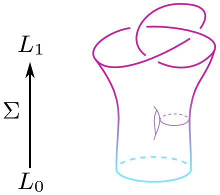

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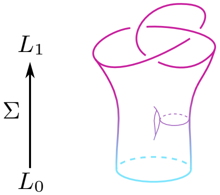

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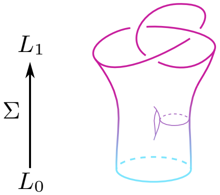

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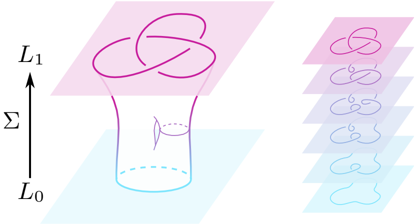

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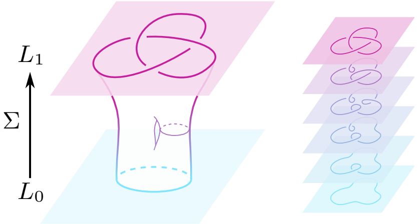

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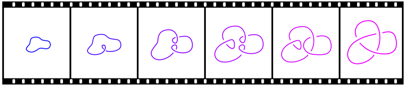

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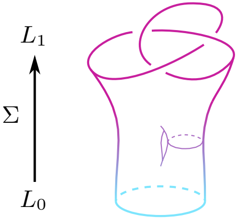			

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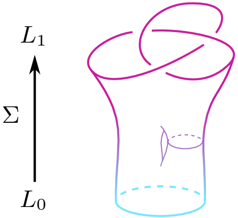			

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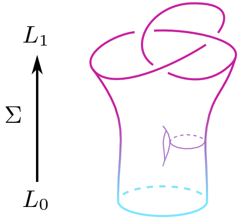
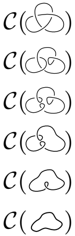
<tr><td>
</td><td></td><td></td><td></td></tr>
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Khovanov homology is a *functor* on the category of link cobordisms.

- links are assigned chain complexes with associated homology groups
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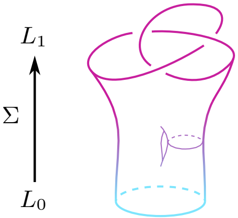		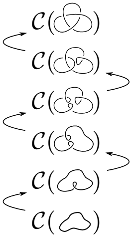	

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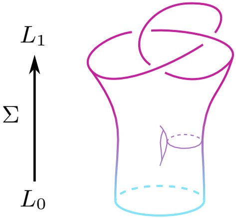
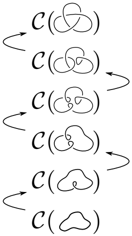
 L_1, Σ, L_0			$\mathcal{C}(D_1)$ ↑ $\mathcal{C}(\Sigma)$ $\mathcal{C}(D_0)$

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Theorem (Khovanov)

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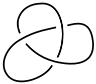

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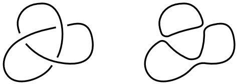

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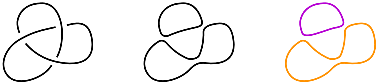

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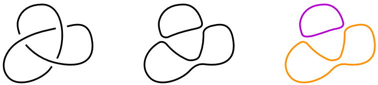

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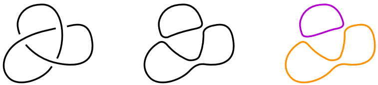

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Let's take a quick look at $\mathcal{C}(3_1)$

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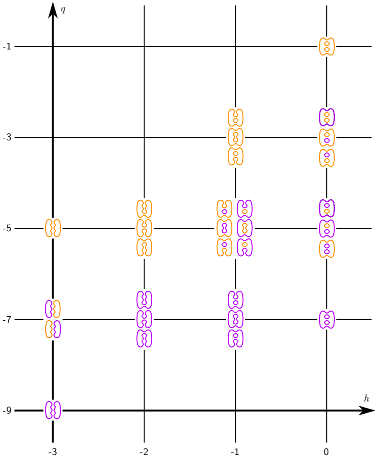

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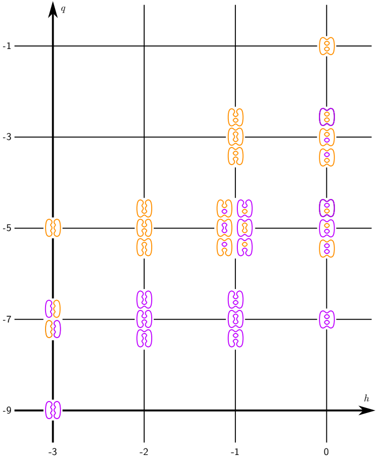

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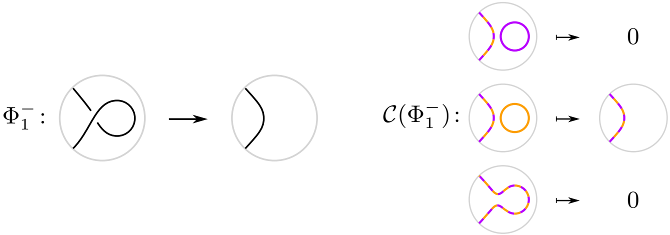

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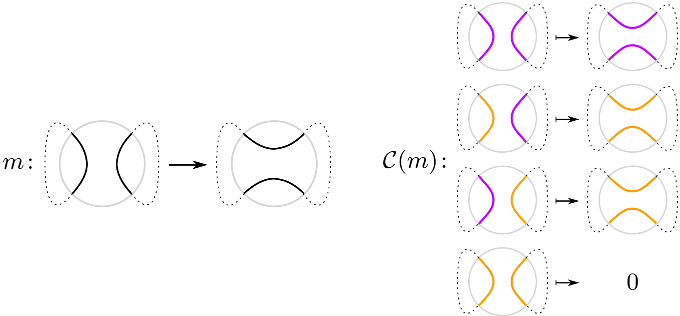

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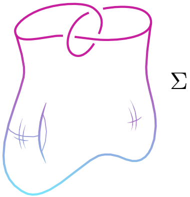

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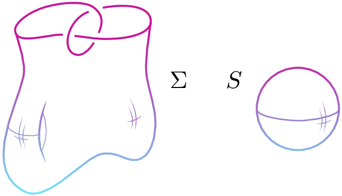

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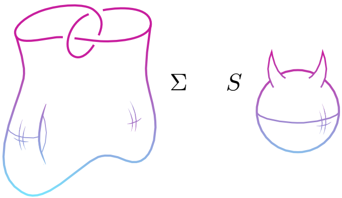

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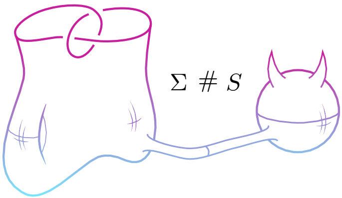

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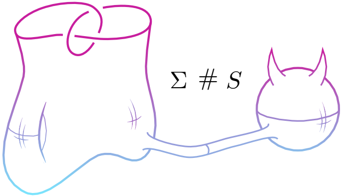

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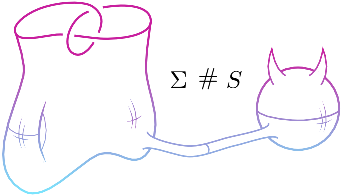

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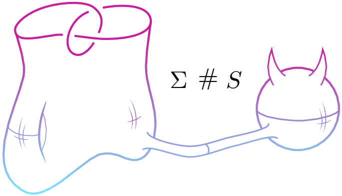

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Takeaway: do not do this when finding Σ, Σ'

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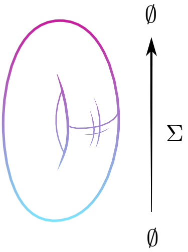

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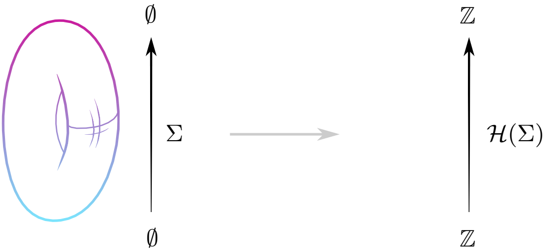

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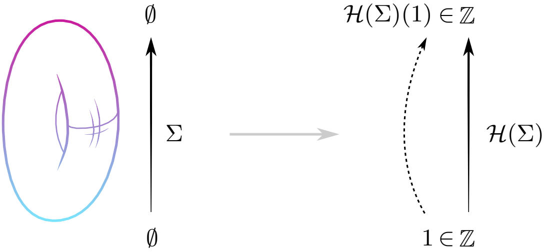

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Theorem (Rasmussen, Tanaka)

The φ-numbers associated to connected $\Sigma \subset B^4$ are determined by genus:

- *if $g(\Sigma) = 1$, then $\varphi(\Sigma) = \pm\,2$*
- *if $g(\Sigma) \neq 1$, then $\varphi(\Sigma) = \ \ 0$*

Cases

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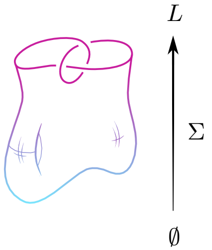

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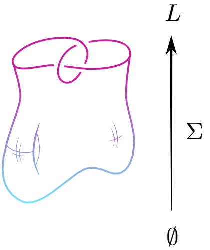

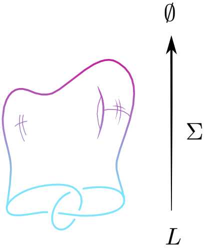

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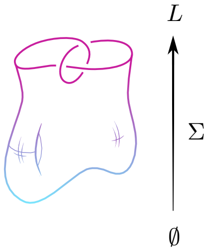

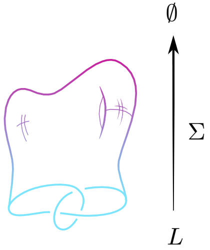

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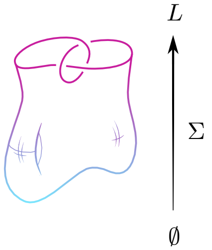

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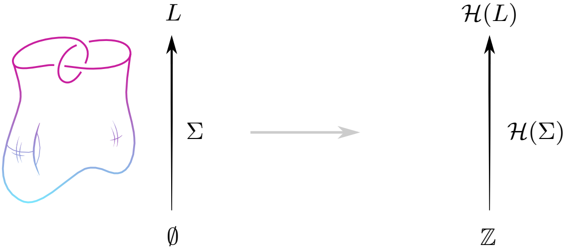

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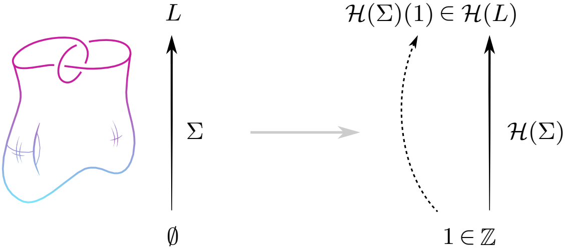

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If so, we say $\Sigma_{0,1}$ are $\boldsymbol{\varphi}$-**distinguished**.

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Theorem (Swann, Sundberg)

The slice disks D_ℓ and D_r for 9_{46} are φ-distinguished, and therefore, are not isotopic rel boundary.

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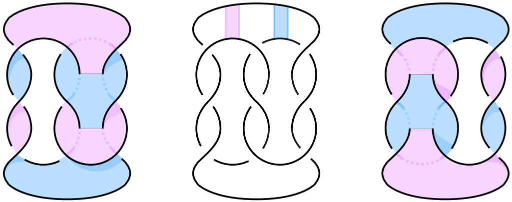

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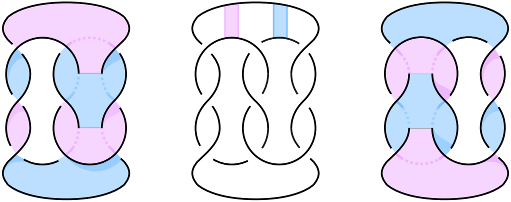

What do $\varphi(D_\ell)$ and $\varphi(D_r)$ look like?

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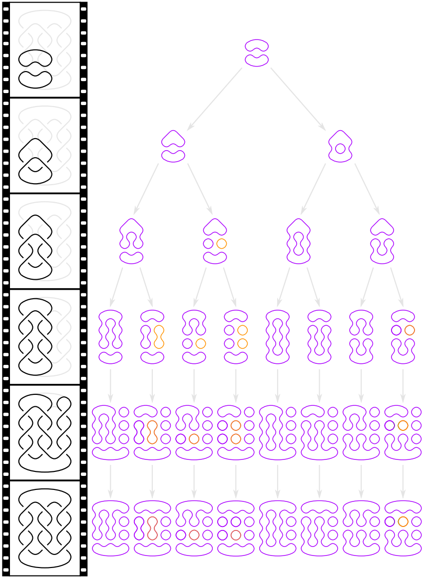

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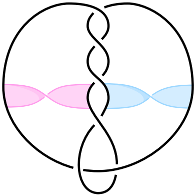

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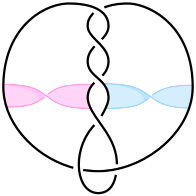

These knots are so nice! Are there even nicer knots out there?

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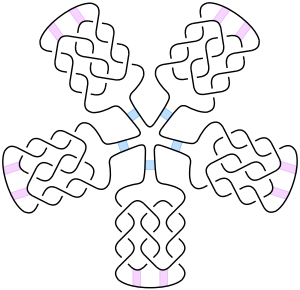
Slice disks are obtained by boundary-summing copies of D_ℓ and D_r.



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- These slice disks are pairwise φ-distinguished using injectivity and functoriality of the induced maps on Khovanov homology:

$$\varphi(C \circ D) = \mathcal{H}(C)(\varphi(D)) \neq \pm\mathcal{H}(C)(\varphi(D')) = \varphi(C \circ D')$$

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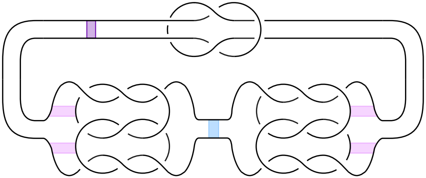

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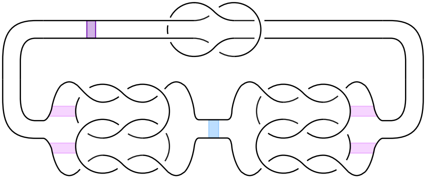

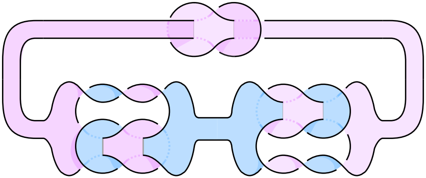

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Example. The φ-class of a genus 1 surface bounding $\mathrm{Wh}_2^+(3_1)$ has approximately 2^{12} smoothings and the matrix representing $h^{-1,1}$ has approximate dimensions $20,000 \times 30,000$

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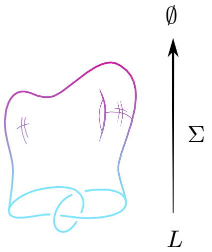

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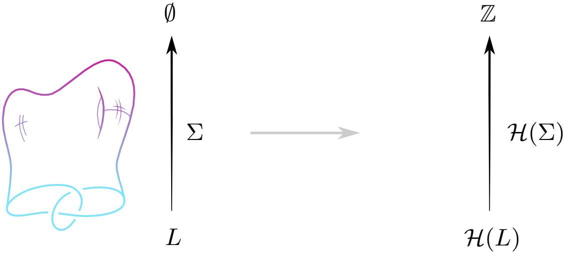

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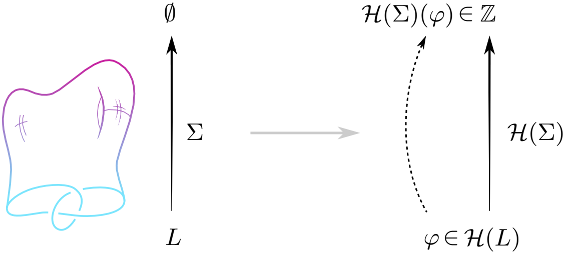

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- it induces a map $\mathcal{H}(\Sigma)\colon \mathcal{H}(L) \to \mathbb{Z}$
- choose a class $\varphi \in \mathcal{H}(L)$, and note that $\mathcal{H}(\Sigma)(\varphi) \in \mathbb{Z}$ is an up-to-sign invariant of the isotopy class of Σ.

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For a link cobordism $\Sigma\colon L \to \emptyset$ and a class $\varphi \in \mathcal{H}(L)$, the **$\varphi^*$-number**

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Lemma

For a link cobordism $\Sigma\colon L \to \emptyset$ and a class $\varphi \in \mathcal{H}(L)$, the **$\varphi^*$-number**

$$\varphi^*(\Sigma) := \mathcal{H}(\Sigma)(\varphi) \in \mathbb{Z}$$

is an up-to-sign invariant of the boundary-preserving isotopy class of Σ.

Do φ^*-numbers distinguish any surfaces with boundary?

Can we find $\Sigma_{0,1} \subset B^4$ bounding a common $L \subset S^3$ *and* a class $\varphi \in \mathcal{H}(L)$ such that $\varphi^*(\Sigma_0) \neq \pm\varphi^*(\Sigma_1)$?

If so, we say $\Sigma_{0,1}$ are **φ^*-distinguished**.

Applications of φ^*-numbers

Theorem (Hayden-Sundberg)

The pair of slice disks D_ℓ and D_r for the knot K (below) are φ^*-distinguished by the given class $\varphi \in \mathcal{H}(K)$, and therefore, are not isotopic rel boundary.

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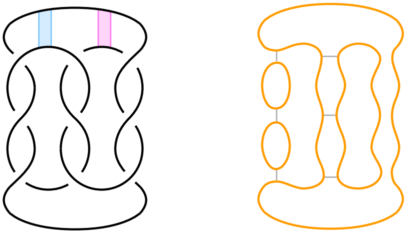

$K = 9_{46}$

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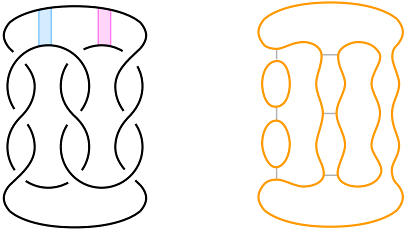

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Proof idea: show $\varphi^*(D_\ell) = 1$ and $\varphi^*(D_r) = 0$

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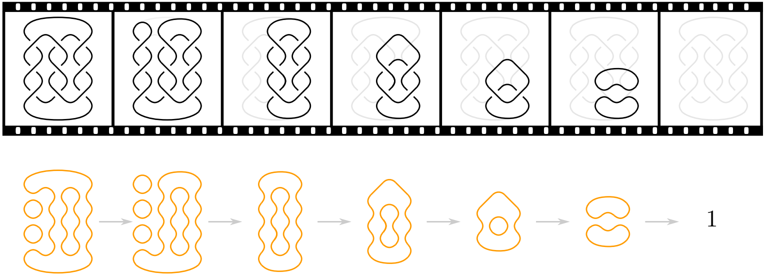

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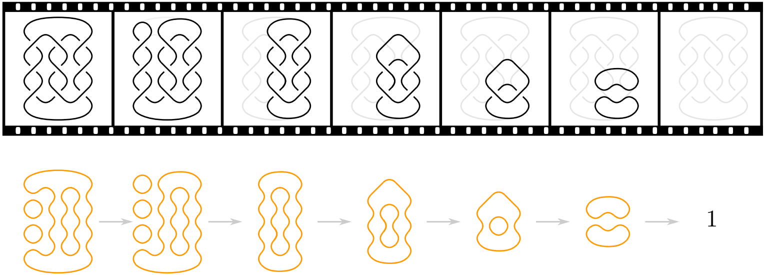

So $\varphi^*(D_\ell) = 1$ and $\varphi^*(D_r) = 0$, as desired.

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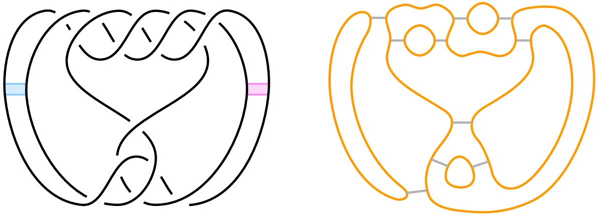

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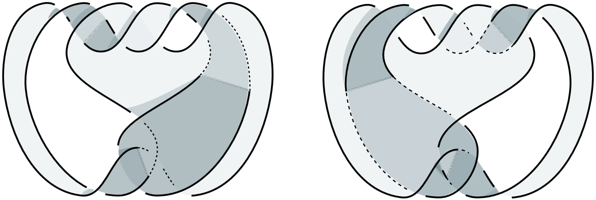

Slice disks for $K = 15n_{103488}$ (image by Kyle Hayden).

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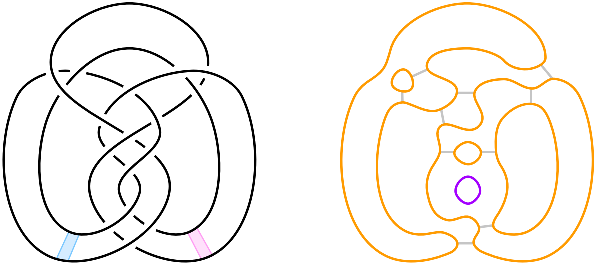

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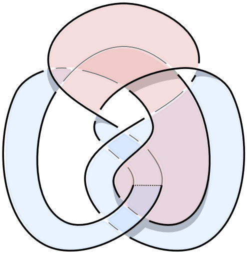

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Bibliography I

- D Bar-Natan, *Khovanov's homology for tangles and cobordisms*, **Geom. Topol.**, 9:1443-1499, 2005.
- Anthony Conway and Mark Powell, *Characterisation of homotopy ribbon discs*, **Adv. Math.**, 391:Paper No. 107960, 2021.
- Kyle Hayden, *Corks, covers, and complex curves*, arXiv:2107.06856, 2021.
- Kyle Hayden and Isaac Sundberg, *Khovanov homology and exotic surfaces in the 4-ball*, arXiv:2108.04810, 2021.
- Magnus Jacobsson, *An invariant of link cobordisms from Khovanov homology*, **Algebr. Geom. Topol.**, 4:1211-1251, 2004.
- András Juhász and Ian Zemke, *Distinguishing slice disks using knot floer homology*, *Seceta Math. (N.S.)*,20(1), 2020.
- Mikhail Khovanov, *A categorification of the Jones polynomial*, **Duke Math. J.**, 101(3):359-426, 2000.

Bibliography II

- Mikhail Khovanov, *An invariant of tangle cobordisms*, **Transactions of the American Mathematical Society**, 358(1):315-327, 2006.
- Adam S. Levine and Ian Zemke, *Khovanov homology and ribbon concordances*, **Bull. Lond. Math. Soc.**, 51(6):1099-1103, 2019.
- Allison N. Miller and Mark Powell, Stabilization distance between surfaces, *Enseign. Math.*,**65**:397-440, 2020.
- Lisa Piccirillo, *The Conway knot is not slice*, **Ann. of Math.** (2), 191(2):581-591, 2020.
- Jacob Rasmussen, *Khovanov's invariant for closed surfaces*, arXiv:math/0502527, 2005.
- Isaac Sundberg and Jonah Swann, *Relative Khovanov-Jacobsson classes*, arXiv:2103.01438, 2021.
- Jonah Swann, *Relative Khovanov-Jacobsson classes of spanning surfaces*, *Ph.D. Thesis, Bryn Mawr College*, 2010.

Bibliography III

Kokoro Tanaka, *Khovanov-Jacobsson numbers and invariants of surface-knots derived from Bar-Natan's theory*, **Proc. Amer. Math. Soc.**, 134(12):3685–3689, 2005.