The Khovanov homology of slice disks

Isaac Sundberg

Collaborators: Jonah Swann & Kyle Hayden

Bryn Mawr College

Columbia Geometry & Topology Seminar

24 September 2021

Table of Contents

1. Motivation
2. Khovanov homology of surfaces
3. Khovanov homology of knotted surfaces
4. Khovanov homology of slice disks
5. Future work

Table of Contents

1 Motivation

2 Khovanov homology of surfaces

3 Khovanov homology of knotted surfaces

4 Khovanov homology of slice disks

5 Future work

Motivation for slice disks

Question:

Motivation for slice disks

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Motivation for slice disks

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Recall: We can view the 3-sphere and 4-ball as follows:

- $S^3 = \mathbb{R}^3 \cup \{\infty\}$
- $B^4 = S^3 \times [0,1]/S^3 \times \{0\}$

Motivation for slice disks

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Recall: We can view the 3-sphere and 4-ball as follows:

- $S^3 = \mathbb{R}^3 \cup \{\infty\}$
- $B^4 = S^3 \times [0,1]/S^3 \times \{0\}$

This allows us to view surfaces $F \subset B^4$ by their level sets $F_i = F \cap (S^3 \times \{i\})$.

Motivation for slice disks

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Recall: We can view the 3-sphere and 4-ball as follows:

- $S^3 = \mathbb{R}^3 \cup \{\infty\}$
- $B^4 = S^3 \times [0,1]/S^3 \times \{0\}$

This allows us to view surfaces $F \subset B^4$ by their level sets $F_i = F \cap (S^3 \times \{i\})$.

Example:

Motivation for slice disks

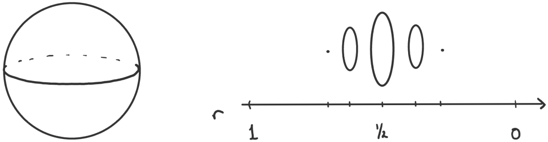
Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Recall: We can view the 3-sphere and 4-ball as follows:

- $S^3 = \mathbb{R}^3 \cup \{\infty\}$
- $B^4 = S^3 \times [0,1]/S^3 \times \{0\}$

This allows us to view surfaces $F \subset B^4$ by their level sets $F_i = F \cap (S^3 \times \{i\})$.

Example: A sphere in the 4-ball might look like:



Motivation for slice disks

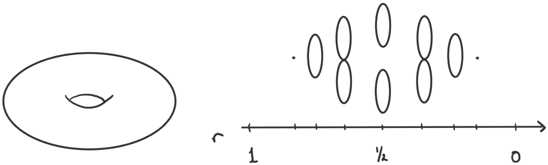
Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Recall: We can view the 3-sphere and 4-ball as follows:

- $S^3 = \mathbb{R}^3 \cup \{\infty\}$
- $B^4 = S^3 \times [0,1] / S^3 \times \{0\}$

This allows us to view surfaces $F \subset B^4$ by their level sets $F_i = F \cap (S^3 \times \{i\})$.

Example: A torus in the 4-ball might look like:



Motivation for slice disks

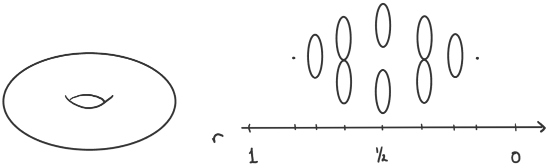
Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Recall: We can view the 3-sphere and 4-ball as follows:

- $S^3 = \mathbb{R}^3 \cup \{\infty\}$
- $B^4 = S^3 \times [0,1]/S^3 \times \{0\}$

This allows us to view surfaces $F \subset B^4$ by their level sets $F_i = F \cap (S^3 \times \{i\})$.

Example: A torus in the 4-ball might look like:



Takeaway: We can answer this question by describing the level sets of a disk D.

Definition of a slice disk

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Definition of a slice disk

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Answer:

Definition of a slice disk

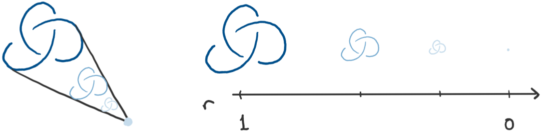
Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Answer: Yes, always!

Definition of a slice disk

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

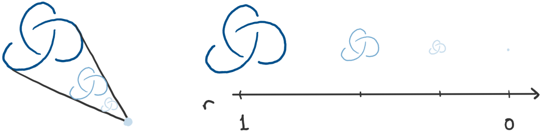
Answer: Yes, always!



Definition of a slice disk

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Answer: Yes, always!

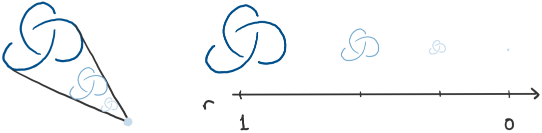


Classic Question:

Definition of a slice disk

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Answer: Yes, always!

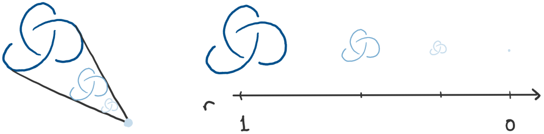


Classic Question:
Given a knot K in the 3-sphere S^3, when does K bound a **smooth** disk D properly embedded in the 4-ball B^4?

Definition of a slice disk

Question:
Given a knot K in the 3-sphere S^3, when does K bound a disk D properly embedded in the 4-ball B^4?

Answer: Yes, always!



Classic Question:
Given a knot K in the 3-sphere S^3, when does K bound a **smooth** disk D properly embedded in the 4-ball B^4?

Definition

A knot $K \subset S^3$ that bounds a smooth, properly embedded disk $D \subset B^4$ is a **slice knot** and D is a **slice disk**.

Example of a slice disk

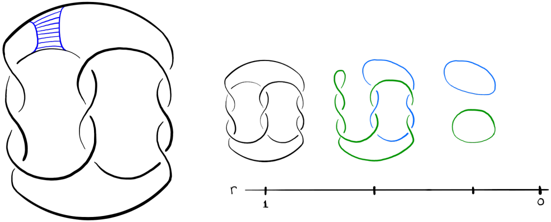
Example:

Example of a slice disk

Example:
The knot 9_{46} is slice, with slice disk D_ℓ described by the following level sets:

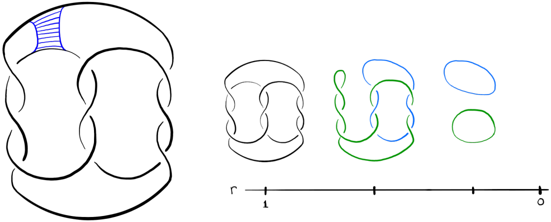
Example of a slice disk

Example:
The knot 9_{46} is slice, with slice disk D_ℓ described by the following level sets:



Example of a slice disk

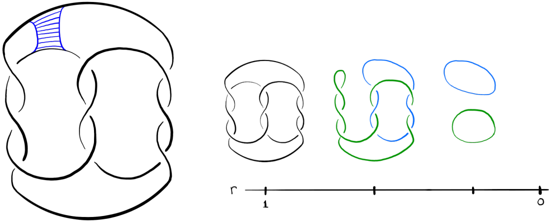
Example:
The knot 9_{46} is slice, with slice disk D_ℓ described by the following level sets:



A second slice D_r can be described similarly, by performing the *band move* on the right-hand-side of 9_{46}.

Example of a slice disk

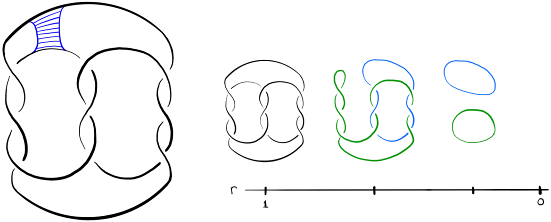
Example:
The knot 9_{46} is slice, with slice disk D_ℓ described by the following level sets:



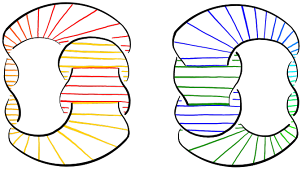
A second slice D_r can be described similarly, by performing the *band move* on the right-hand-side of 9_{46}. We can se these disks pushed into S^3 as:

Example of a slice disk

Example:
The knot 9_{46} is slice, with slice disk D_ℓ described by the following level sets:



A second slice D_r can be described similarly, by performing the *band move* on the right-hand-side of 9_{46}. We can se these disks pushed into S^3 as:



Equivalence of slice disks

Follow-up Question:

Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

Equivalence of slice disks

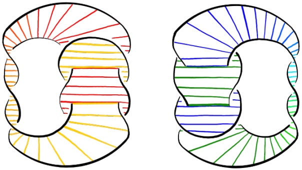
Follow-up Question:
Are D_ℓ and D_r isotopic?

Answer:

Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

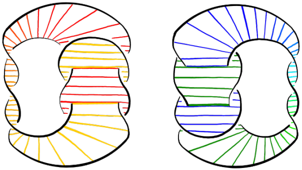
Answer:
Yes - by a rotation!



Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

Answer:
Yes - by a rotation!

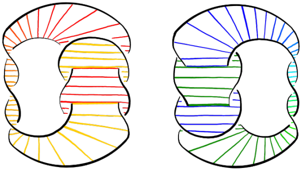


Follow-up Question:

Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

Answer:
Yes - by a rotation!

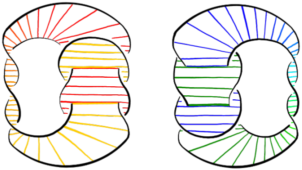


Follow-up Question:
Are D_ℓ and D_r isotopic rel boundary (i.e. leaving 9_{46} fixed)?

Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

Answer:
Yes - by a rotation!



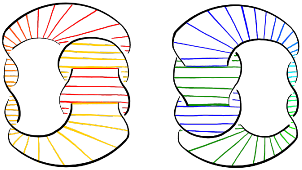
Follow-up Question:
Are D_ℓ and D_r isotopic rel boundary (i.e. leaving 9_{46} fixed)?

Answer:

Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

Answer:
Yes - by a rotation!



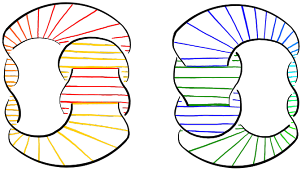
Follow-up Question:
Are D_ℓ and D_r isotopic rel boundary (i.e. leaving 9_{46} fixed)?

Answer:
Maybe? Not exactly easy to tell without doing some math...

Equivalence of slice disks

Follow-up Question:
Are D_ℓ and D_r isotopic?

Answer:
Yes - by a rotation!



Follow-up Question:
Are D_ℓ and D_r isotopic rel boundary (i.e. leaving 9_{46} fixed)?

Answer:
Maybe? Not exactly easy to tell without doing some math...

We need techniques for studying surfaces up to boundary-preserving isotopy!

Methods for studying slice disks

There are multiple ways to study slice disks up to boundary-preserving isotopy:

Methods for studying slice disks

There are multiple ways to study slice disks up to boundary-preserving isotopy:

- Fundamental group of the compliment

Methods for studying slice disks

There are multiple ways to study slice disks up to boundary-preserving isotopy:

- Fundamental group of the compliment
- Alexander modules

Methods for studying slice disks

There are multiple ways to study slice disks up to boundary-preserving isotopy:

- Fundamental group of the compliment
- Alexander modules
- gauge theory

Methods for studying slice disks

There are multiple ways to study slice disks up to boundary-preserving isotopy:

- Fundamental group of the compliment
- Alexander modules
- gauge theory
- knot Floer homology

Methods for studying slice disks

There are multiple ways to study slice disks up to boundary-preserving isotopy:

- Fundamental group of the compliment
- Alexander modules
- gauge theory
- knot Floer homology
- Khovanov homology

Table of Contents

1 Motivation

2 Khovanov homology of surfaces

3 Khovanov homology of knotted surfaces

4 Khovanov homology of slice disks

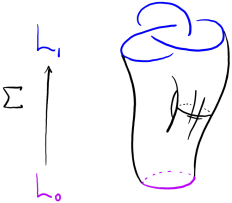
5 Future work

Link cobordisms

Definition. A **link cobordism** $\Sigma\colon L_0 \to L_1$ is a smooth, compact, oriented, properly embedded surface $\Sigma \subset S^3 \times [0,1]$ with boundary a pair ($i \in \{0,1\}$) of oriented links $L_i = \Sigma \cap (S^3 \times \{i\})$.

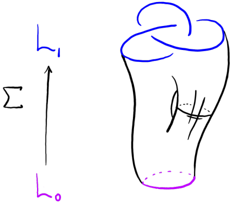
Link cobordisms

Definition. A **link cobordism** $\Sigma\colon L_0 \to L_1$ is a smooth, compact, oriented, properly embedded surface $\Sigma \subset S^3 \times [0,1]$ with boundary a pair ($i \in \{0,1\}$) of oriented links $L_i = \Sigma \cap (S^3 \times \{i\})$.



Link cobordisms

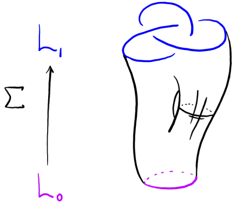
Definition. A **link cobordism** $\Sigma\colon L_0 \to L_1$ is a smooth, compact, oriented, properly embedded surface $\Sigma \subset S^3 \times [0,1]$ with boundary a pair ($i \in \{0,1\}$) of oriented links $L_i = \Sigma \cap (S^3 \times \{i\})$.



Examples: slices ($\emptyset \to K$), closed surfaces ($\emptyset \to \emptyset$), Seifert surfaces ($\emptyset \to K$)

Link cobordisms

Definition. A **link cobordism** $\Sigma\colon L_0 \to L_1$ is a smooth, compact, oriented, properly embedded surface $\Sigma \subset S^3 \times [0,1]$ with boundary a pair ($i \in \{0,1\}$) of oriented links $L_i = \Sigma \cap (S^3 \times \{i\})$.

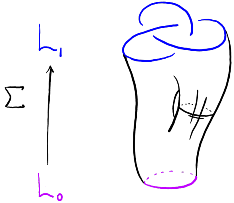


Examples: slices ($\emptyset \to K$), closed surfaces ($\emptyset \to \emptyset$), Seifert surfaces ($\emptyset \to K$)

Definition. A link cobordism $\Sigma\colon L_0 \to L_1$ can be represented as a **movie**: a finite sequence of diagrams $\{D_{t_i}\}_{i=0}^{n}$, with each successive pair related by an isotopy, Morse move, or Reidemeister move.

Link cobordisms

Definition. A **link cobordism** $\Sigma\colon L_0 \to L_1$ is a smooth, compact, oriented, properly embedded surface $\Sigma \subset S^3 \times [0,1]$ with boundary a pair ($i \in \{0,1\}$) of oriented links $L_i = \Sigma \cap (S^3 \times \{i\})$.



Examples: slices ($\emptyset \to K$), closed surfaces ($\emptyset \to \emptyset$), Seifert surfaces ($\emptyset \to K$)

Definition. A link cobordism $\Sigma\colon L_0 \to L_1$ can be represented as a **movie**: a finite sequence of diagrams $\{D_{t_i}\}_{i=0}^{n}$, with each successive pair related by an isotopy, Morse move, or Reidemeister move.

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathsf{Kh}(D)$ with homology $\mathsf{Kh}(D)$, called the Khovanov homology.

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathsf{Kh}(D)$ with homology $\mathsf{Kh}(D)$, called the Khovanov homology.

How do we define this chain complex?

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathrm{Kh}(D)$ with homology $\mathrm{Kh}(D)$, called the Khovanov homology.

How do we define this chain complex?

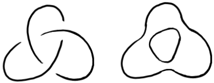
- Choose a diagram D for your link

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathrm{Kh}(D)$ with homology $\mathrm{Kh}(D)$, called the Khovanov homology.

How do we define this chain complex?



- Choose a diagram D for your link
- Smooth each crossing $\times$ in D as a 0-smoothing $\asymp$ or a 1-smoothing $)($

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathrm{Kh}(D)$ with homology $\mathrm{Kh}(D)$, called the Khovanov homology.

How do we define this chain complex?

- Choose a diagram D for your link
- Smooth each crossing in D as a 0-smoothing or a 1-smoothing
- Label each resulting component with a 1 or an x

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathrm{Kh}(D)$ with homology $\mathrm{Kh}(D)$, called the Khovanov homology.

How do we define this chain complex?

- Choose a diagram D for your link
- Smooth each crossing in D as a 0-smoothing or a 1-smoothing
- Label each resulting component with a 1 or an x
- Generate $\mathcal{C}\mathrm{Kh}(D)$ over $\mathbb{Z}$ with all possible *labeled smoothings*

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathsf{Kh}(D)$ with homology $\mathsf{Kh}(D)$, called the Khovanov homology.

Properties:

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathsf{Kh}(D)$ with homology $\mathsf{Kh}(D)$, called the Khovanov homology.

Properties:

- Different diagrams have isomorphic Khovanov homology
  (we write $\mathsf{Kh}(L)$ to mean: choose a diagram D for L and consider $\mathsf{Kh}(D)$)

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathsf{Kh}(D)$ with homology $\mathsf{Kh}(D)$, called the Khovanov homology.

Properties:

- Different diagrams have isomorphic Khovanov homology (we write $\mathsf{Kh}(L)$ to mean: choose a diagram D for L and consider $\mathsf{Kh}(D)$)
- We set $\mathcal{C}\mathsf{Kh}(\emptyset) = \mathbb{Z}$ and $\mathsf{Kh}(\emptyset) = \mathbb{Z}$

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathsf{Kh}(D)$ with homology $\mathsf{Kh}(D)$, called the Khovanov homology.

Properties:

- Different diagrams have isomorphic Khovanov homology (we write $\mathsf{Kh}(L)$ to mean: choose a diagram D for L and consider $\mathsf{Kh}(D)$)
- We set $\mathcal{C}\mathsf{Kh}(\emptyset) = \mathbb{Z}$ and $\mathsf{Kh}(\emptyset) = \mathbb{Z}$
- There is a bigrading $\mathcal{C}\mathsf{Kh}^{h,q}(D)$

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathsf{Kh}(D)$ with homology $\mathsf{Kh}(D)$, called the Khovanov homology.

Properties:

- Different diagrams have isomorphic Khovanov homology (we write $\mathsf{Kh}(L)$ to mean: choose a diagram D for L and consider $\mathsf{Kh}(D)$)
- We set $\mathcal{C}\mathsf{Kh}(\emptyset) = \mathbb{Z}$ and $\mathsf{Kh}(\emptyset) = \mathbb{Z}$
- There is a bigrading $\mathcal{C}\mathsf{Kh}^{h,q}(D)$
- There is a differential $d\colon \mathcal{C}\mathsf{Kh}^{h,q}(D) \to \mathcal{C}\mathsf{Kh}^{h+1,q}(D)$

Khovanov homology of links

Theorem (Khovanov '00)

A diagram D of an oriented link L induces a chain complex $\mathcal{C}\mathsf{Kh}(D)$ with homology $\mathsf{Kh}(D)$, called the Khovanov homology.

Properties:

- Different diagrams have isomorphic Khovanov homology (we write $\mathsf{Kh}(L)$ to mean: choose a diagram D for L and consider $\mathsf{Kh}(D)$)
- We set $\mathcal{C}\mathsf{Kh}(\emptyset) = \mathbb{Z}$ and $\mathsf{Kh}(\emptyset) = \mathbb{Z}$
- There is a bigrading $\mathcal{C}\mathsf{Kh}^{h,q}(D)$
- There is a differential $d\colon \mathcal{C}\mathsf{Kh}^{h,q}(D) \to \mathcal{C}\mathsf{Kh}^{h+1,q}(D)$
- Many similarly defined *link homology theories* exist

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^n$ of a link cobordism $\Sigma\colon L_0 \to L_1$ induces a chain map

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^n$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

How do we define these chain maps?

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

How do we define these chain maps?

- The diagrams D_{t_i} in the movie each have an associated chain complex

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

How do we define these chain maps?

- The diagrams D_{t_i} in the movie each have an associated chain complex
- Adjacent frames $D_{t_i} \to D_{t_{i+1}}$ are related by an isotopy, Morse move, or Reidemeister moves

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ of a link cobordism $\Sigma\colon L_0 \to L_1$ induces a chain map

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

How do we define these chain maps?

- The diagrams D_{t_i} in the movie each have an associated chain complex
- Adjacent frames $D_{t_i} \to D_{t_{i+1}}$ are related by an isotopy, Morse move, or Reidemeister moves
- Define chain maps for each of these moves

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ of a link cobordism $\Sigma\colon L_0 \to L_1$ induces a chain map

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

How do we define these chain maps?

- The diagrams D_{t_i} in the movie each have an associated chain complex
- Adjacent frames $D_{t_i} \to D_{t_{i+1}}$ are related by an isotopy, Morse move, or Reidemeister moves
- Define chain maps for each of these moves
- Compose these chain maps to produce $\mathcal{C}\mathsf{Kh}(\Sigma)$

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

Properties:

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ *of a link cobordism* $\Sigma\colon L_0 \to L_1$ *induces a chain map*

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

Properties:

- This map is also bigraded:

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}^{h,q}(D_0) \to \mathcal{C}\mathsf{Kh}^{h,q+\chi(\Sigma)}(D_1)$$

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ of a link cobordism $\Sigma\colon L_0 \to L_1$ induces a chain map

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ on homology.

Properties:

- This map is also bigraded:

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}^{h,q}(D_0) \to \mathcal{C}\mathsf{Kh}^{h,q+\chi(\Sigma)}(D_1)$$

- Generally, they are difficult to compute...

Khovanov homology of surfaces

Theorem (Khovanov '00)

A movie $\{D_{t_i}\}_{i=0}^{n}$ of a link cobordism $\Sigma\colon L_0 \to L_1$ induces a chain map

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}(D_0) \to \mathcal{C}\mathsf{Kh}(D_1)$$

with induced homomorphism $\mathsf{Kh}(\Sigma)$ *on homology.*

Properties:

- This map is also bigraded:

$$\mathcal{C}\mathsf{Kh}(\Sigma)\colon \mathcal{C}\mathsf{Kh}^{h,q}(D_0) \to \mathcal{C}\mathsf{Kh}^{h,q+\chi(\Sigma)}(D_1)$$

- Generally, they are difficult to compute...
- But they have one very useful property!

Invariance

Theorem (Jacobsson '04, Bar-Natan '05, Khovanov '06)

The map on Khovanov homology induced by a link cobordism Σ is invariant, up to sign, under smooth boundary-preserving isotopy of Σ.

Invariance

Theorem (Jacobsson '04, Bar-Natan '05, Khovanov '06)

The map on Khovanov homology induced by a link cobordism Σ is invariant, up to sign, under smooth boundary-preserving isotopy of Σ.

$$\Sigma \simeq_\partial \Sigma' \implies \mathsf{Kh}(\Sigma) = \pm\mathsf{Kh}(\Sigma')$$

Invariance

Theorem (Jacobsson '04, Bar-Natan '05, Khovanov '06)

The map on Khovanov homology induced by a link cobordism Σ is invariant, up to sign, under smooth boundary-preserving isotopy of Σ.

$$\Sigma \simeq_\partial \Sigma' \implies \mathsf{Kh}(\Sigma) = \pm \mathsf{Kh}(\Sigma')$$

Goal:

Invariance

Theorem (Jacobsson '04, Bar-Natan '05, Khovanov '06)

The map on Khovanov homology induced by a link cobordism Σ is invariant, up to sign, under smooth boundary-preserving isotopy of Σ.

$$\Sigma \simeq_\partial \Sigma' \implies \mathsf{Kh}(\Sigma) = \pm\mathsf{Kh}(\Sigma')$$

Goal:
Distinguish link cobordisms Σ, Σ' up to **smooth** isotopy rel boundary by showing their induced maps
are distinct $\mathsf{Kh}(\Sigma) \neq \pm\mathsf{Kh}(\Sigma')$

Table of Contents

1 Motivation

2 Khovanov homology of surfaces

3 Khovanov homology of knotted surfaces

4 Khovanov homology of slice disks

5 Future work

Khovanov-Jacobsson numbers

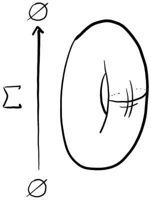
Question:

Khovanov-Jacobsson numbers

Question:
Can these induced maps distinguish (closed) knotted surfaces in B^4?

Khovanov-Jacobsson numbers

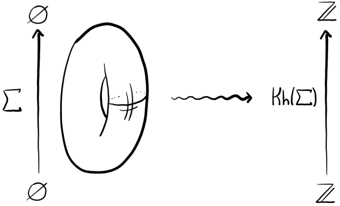
Question:
Can these induced maps distinguish (closed) knotted surfaces in B^4?



- A knotted surface Σ can be regarded as a link cobordism $\Sigma\colon \emptyset \to \emptyset$.

Khovanov-Jacobsson numbers

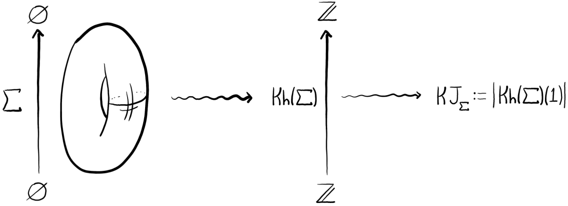
Question:
Can these induced maps distinguish (closed) knotted surfaces in B^4?



- A knotted surface Σ can be regarded as a link cobordism $\Sigma\colon \emptyset \to \emptyset$.
- It induces a map $\mathrm{Kh}(\Sigma)\colon \mathbb{Z} \to \mathbb{Z}$

Khovanov-Jacobsson numbers

Question:
Can these induced maps distinguish (closed) knotted surfaces in B^4?



- A knotted surface Σ can be regarded as a link cobordism $\Sigma\colon \emptyset \to \emptyset$.
- It induces a map $\mathsf{Kh}(\Sigma)\colon \mathbb{Z} \to \mathbb{Z}$
- This map is determined by $\mathsf{Kh}(\Sigma)(1) \in \mathbb{Z}$, so this integer is an up-to-sign invariant of the (ambient) isotopy class of Σ

Rasmussen-Tanaka

Lemma

For a link cobordism $\Sigma\colon \emptyset \to \emptyset$, the Khovanov-Jacobsson number

$$\mathsf{KJ}_\Sigma := |\mathsf{Kh}(\Sigma)(1)| \in \mathbb{Z}$$

is an invariant of the ambient isotopy class of Σ.

Rasmussen-Tanaka

Lemma

For a link cobordism $\Sigma \colon \emptyset \to \emptyset$, the Khovanov-Jacobsson number

$$\mathsf{KJ}_\Sigma := |\mathsf{Kh}(\Sigma)(1)| \in \mathbb{Z}$$

is an invariant of the ambient isotopy class of Σ.

Question. Do Khovanov-Jacobsson numbers distinguish any knotted surfaces?

Rasmussen-Tanaka

Lemma

For a link cobordism $\Sigma\colon \emptyset \to \emptyset$, the Khovanov-Jacobsson number

$$\mathsf{KJ}_\Sigma := |\mathsf{Kh}(\Sigma)(1)| \in \mathbb{Z}$$

is an invariant of the ambient isotopy class of Σ.

Question. Do Khovanov-Jacobsson numbers distinguish any knotted surfaces?

Theorem (Rasmussen '05, Tanaka '05)

Khovanov-Jacobsson numbers of connected Σ are determined by genus:

- *if $g(\Sigma) = 1$, then $\mathsf{KJ}_\Sigma = 2$*
- *if $g(\Sigma) \neq 1$, then $\mathsf{KJ}_\Sigma = 0$*

Table of Contents

1 Motivation

2 Khovanov homology of surfaces

3 Khovanov homology of knotted surfaces

4 Khovanov homology of slice disks

5 Future work

Cases

Idea:

Cases

Idea:
Follow the same procedure for surfaces with boundary.

Cases

Idea:
Follow the same procedure for surfaces with boundary.

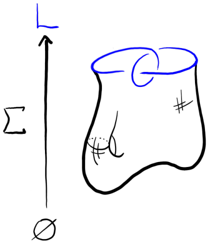
A (nice) surface $\Sigma \subset B^4$ with boundary $L \subset S^3$ can be regarded as:

Cases

Idea:
Follow the same procedure for surfaces with boundary.

A (nice) surface $\Sigma \subset B^4$ with boundary $L \subset S^3$ can be regarded as:

a. a link cobordism $\Sigma\colon \emptyset \to L$, or



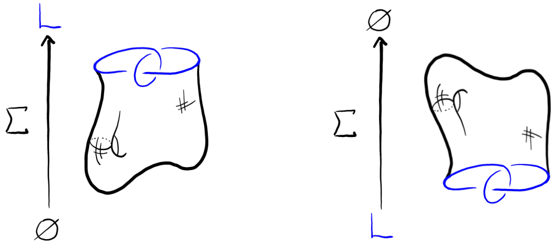
Cases

Idea:
Follow the same procedure for surfaces with boundary.

A (nice) surface $\Sigma \subset B^4$ with boundary $L \subset S^3$ can be regarded as:

a. a link cobordism $\Sigma \colon \emptyset \to L$, or

b. its reverse cobordism $\Sigma \colon L \to \emptyset$



Cases

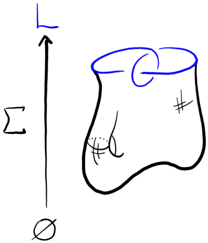
Idea:
Follow the same procedure for surfaces with boundary.

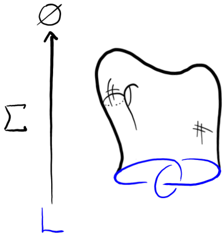
A (nice) surface $\Sigma \subset B^4$ with boundary $L \subset S^3$ can be regarded as:

a. a link cobordism $\Sigma\colon \emptyset \to L$, or

b. its reverse cobordism $\Sigma\colon L \to \emptyset$

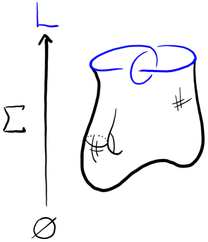
We consider these cases separately.





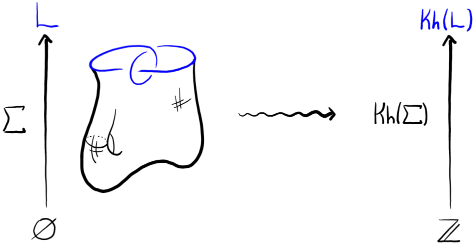
Khovanov-Jacobsson classes

Case 1: Consider a link cobordism $\Sigma\colon \emptyset \to L$



Khovanov-Jacobsson classes

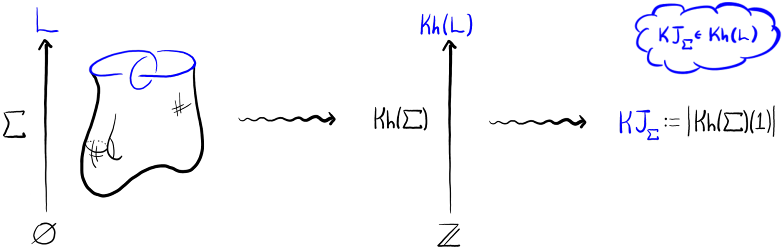
Case 1: Consider a link cobordism $\Sigma\colon \emptyset \to L$



Consider the induced map $\mathsf{Kh}(\Sigma)\colon \mathbb{Z} \to \mathsf{Kh}(L)$

Khovanov-Jacobsson classes

Case 1: Consider a link cobordism $\Sigma\colon \emptyset \to L$



Consider the induced map $\mathsf{Kh}(\Sigma)\colon \mathbb{Z} \to \mathsf{Kh}(L)$

This map is determined by $\mathsf{Kh}(\Sigma)(1) \in \mathsf{Kh}(L)$, so this homology class is an up-to-sign invariant of the (relative) isotopy class of Σ.

Khovanov-Jacobsson classes

Case 1: Consider a link cobordism $\Sigma\colon \emptyset \to L$

Lemma

For a link cobordism $\Sigma\colon \emptyset \to L$, the Khovanov-Jacobsson class

$$\mathsf{KJ}_\Sigma := |\mathsf{Kh}(\Sigma)(1)| \in \mathsf{Kh}(L)$$

is an invariant of the boundary-preserving isotopy class of Σ.

Khovanov-Jacobsson classes

Case 1: Consider a link cobordism $\Sigma\colon \emptyset \to L$

Lemma

For a link cobordism $\Sigma\colon \emptyset \to L$, the Khovanov-Jacobsson class

$$\mathsf{KJ}_\Sigma := |\mathsf{Kh}(\Sigma)(1)| \in \mathsf{Kh}(L)$$

is an invariant of the boundary-preserving isotopy class of Σ.

Question:
Do Khovanov-Jacobsson classes distinguish any surfaces?

Khovanov-Jacobsson classes

Case 1: Consider a link cobordism $\Sigma\colon \emptyset \to L$

Lemma

For a link cobordism $\Sigma\colon \emptyset \to L$, the Khovanov-Jacobsson class

$$\mathsf{KJ}_\Sigma := |\mathsf{Kh}(\Sigma)(1)| \in \mathsf{Kh}(L)$$

is an invariant of the boundary-preserving isotopy class of Σ.

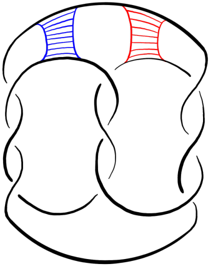
Question:
Do Khovanov-Jacobsson classes distinguish any surfaces?

Hopefully!

Khovanov-Jacobsson classes

Theorem (Swann '10)

The slice disks D_ℓ and D_r for 9_{46} have distinct Khovanov-Jacobsson classes $\mathsf{KJ}_{D_\ell} \neq \mathsf{KJ}_{D_r}$*, and therefore, are not isotopic rel boundary.*



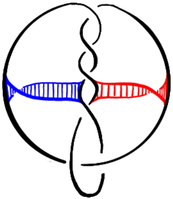
Khovanov-Jacobsson classes

Theorem (Swann '10)

The slice disks D_ℓ and D_r for 9_{46} have distinct Khovanov-Jacobsson classes $\mathsf{KJ}_{D_\ell} \neq \mathsf{KJ}_{D_r}$*, and therefore, are not isotopic rel boundary.*

Theorem (S. '20)

The slice disks D_ℓ and D_r for 6_1 (below) have distinct Khovanov-Jacobsson classes $\mathsf{KJ}_{D_\ell} \neq \mathsf{KJ}_{D_r}$*, and therefore, are not isotopic rel boundary.*



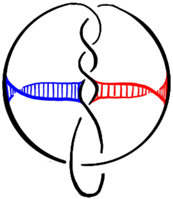
Khovanov-Jacobsson classes

Theorem (Swann '10)

The slice disks D_ℓ and D_r for 9_{46} have distinct Khovanov-Jacobsson classes $\mathsf{KJ}_{D_\ell} \neq \mathsf{KJ}_{D_r}$, and therefore, are not isotopic rel boundary.

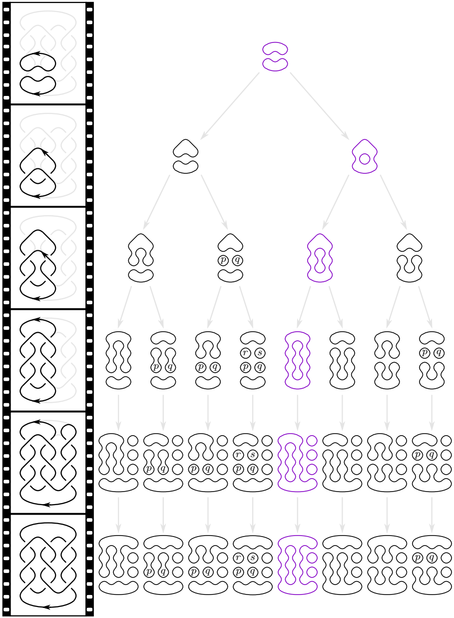
Theorem (S. '20)

The slice disks D_ℓ and D_r for 6_1 (below) have distinct Khovanov-Jacobsson classes $\mathsf{KJ}_{D_\ell} \neq \mathsf{KJ}_{D_r}$, and therefore, are not isotopic rel boundary.



Note: this uniqueness is also known through other techniques.

Calculation for 9_{46}



Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

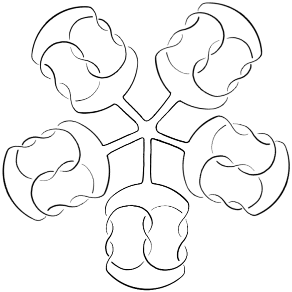
The 2^n slices of $\#_n(9_{46})$ have distinct Khovanov-Jacobsson classes, and therefore, are not isotopic rel boundary.

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

The 2^n slices of $\#_n(9_{46})$ have distinct Khovanov-Jacobsson classes, and therefore, are not isotopic rel boundary.

Slices are obtained by choosing one of the band moves for each copy of 9_{46} (or boundary connect summing the slices).

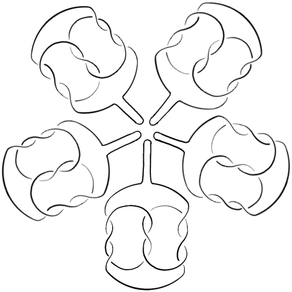


Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

The 2^n slices of $\#_n(9_{46})$ have distinct Khovanov-Jacobsson classes, and therefore, are not isotopic rel boundary.

Slices are obtained by choosing one of the band moves for each copy of 9_{46} (or boundary connect summing the slices).

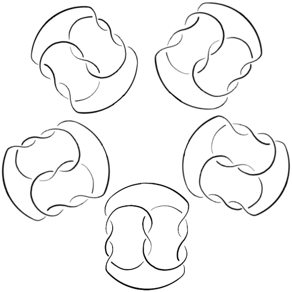


Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

The 2^n slices of $\#_n(9_{46})$ have distinct Khovanov-Jacobsson classes, and therefore, are not isotopic rel boundary.

Slices are obtained by choosing one of the band moves for each copy of 9_{46} (or boundary connect summing the slices).

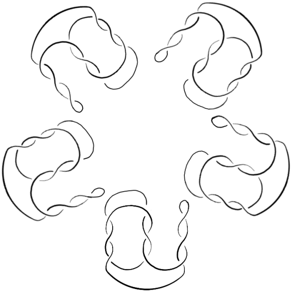


Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

The 2^n slices of $\#_n(9_{46})$ have distinct Khovanov-Jacobsson classes, and therefore, are not isotopic rel boundary.

Slices are obtained by choosing one of the band moves for each copy of 9_{46} (or boundary connect summing the slices).

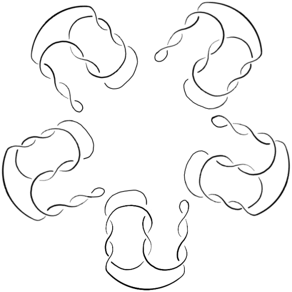


Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

The 2^n slices of $\#_n(9_{46})$ have distinct Khovanov-Jacobsson classes, and therefore, are not isotopic rel boundary.

Slices are obtained by choosing one of the band moves for each copy of 9_{46} (or boundary connect summing the slices).



This can also be done with $\#_n(6_1)$, or even by using combinations of 9_{46} and 6_1.

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

- Every knot is ribbon concordant to a prime knot

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

- Every knot is ribbon concordant to a prime knot
- Ribbon concordances induce injections on Khovanov homology

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

- Every knot is ribbon concordant to a prime knot
- Ribbon concordances induce injections on Khovanov homology
- So, extend the 2^n slices for $\#_n(9_{46})$ by a ribbon-concordance to a prime knot

Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

- Every knot is ribbon concordant to a prime knot
- Ribbon concordances induce injections on Khovanov homology
- So, extend the 2^n slices for $\#_n(9_{46})$ by a ribbon-concordance to a prime knot
- Slices will continue to have distinct Khovanov-Jacobsson classes

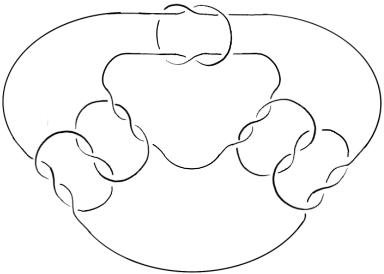
Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

- Every knot is ribbon concordant to a prime knot
- Ribbon concordances induce injections on Khovanov homology
- So, extend the 2^n slices for $\#_n(9_{46})$ by a ribbon-concordance to a prime knot
- Slices will continue to have distinct Khovanov-Jacobsson classes



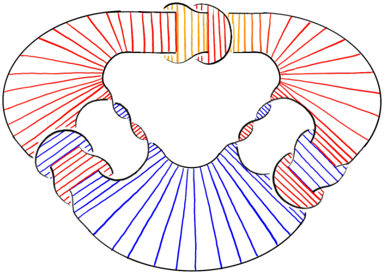
Khovanov-Jacobsson classes

Theorem (S.-Swann '21)

There are prime knots with 2^n slices having distinct Khovanov-Jacobsson classes, and therefore, they are not isotopic rel boundary.

Idea:

- Every knot is ribbon concordant to a prime knot
- Ribbon concordances induce injections on Khovanov homology
- So, extend the 2^n slices for $\#_n(9_{46})$ by a ribbon-concordance to a prime knot
- Slices will continue to have distinct Khovanov-Jacobsson classes



Are we still on case 1?

Downside to Khovanov-Jacobsson classes:

Are we still on case 1?

Downside to Khovanov-Jacobsson classes:

- Hard to calculate...

Are we still on case 1?

Downside to Khovanov-Jacobsson classes:

- Hard to calculate...
- Hard to distinguish...

Are we still on case 1?

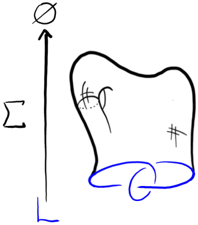
Downside to Khovanov-Jacobsson classes:

- Hard to calculate...
- Hard to distinguish...

Is there a better way?

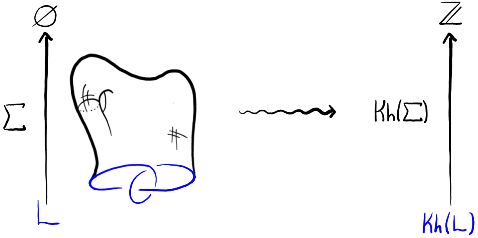
Reverse

Case 2: Consider a link cobordism $\Sigma \colon L \to \emptyset$



Reverse

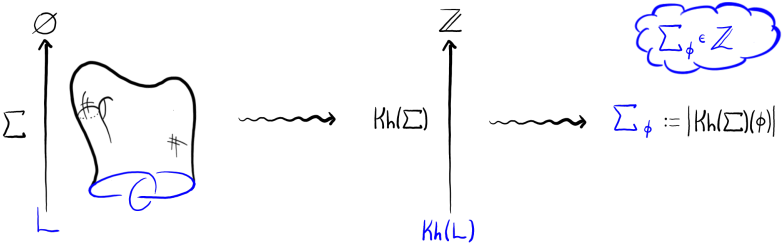
Case 2: Consider a link cobordism $\Sigma\colon L \to \emptyset$



Consider the induced map $\mathsf{Kh}(\Sigma)\colon \mathsf{Kh}(L) \to \mathbb{Z}$

Reverse

Case 2: Consider a link cobordism $\Sigma \colon L \to \emptyset$



Consider the induced map $\mathsf{Kh}(\Sigma) \colon \mathsf{Kh}(L) \to \mathbb{Z}$

Choose a class $\phi \in \mathsf{Kh}(L)$, and note that $\mathsf{Kh}(\Sigma)(\phi) \in \mathbb{Z}$ is an up-to-sign invariant of the (relative) isotopy class of Σ.

Reverse

Case 2: Consider a link cobordism $\Sigma\colon L \to \emptyset$

Lemma

For a link cobordism $\Sigma\colon L \to \emptyset$ and a class $\phi \in \mathsf{Kh}(L)$, the integer

$$\Sigma_\phi := |\mathsf{Kh}(\Sigma)(\phi)| \in \mathbb{Z}$$

is an invariant of the boundary-preserving isotopy class of Σ.

Reverse

Case 2: Consider a link cobordism $\Sigma \colon L \to \emptyset$

Lemma

For a link cobordism $\Sigma \colon L \to \emptyset$ and a class $\phi \in \mathsf{Kh}(L)$, the integer

$$\Sigma_\phi := |\mathsf{Kh}(\Sigma)(\phi)| \in \mathbb{Z}$$

is an invariant of the boundary-preserving isotopy class of Σ.

Questions:
Do these invariants distinguish any surfaces?

Reverse

Case 2: Consider a link cobordism $\Sigma\colon L \to \emptyset$

Lemma

For a link cobordism $\Sigma\colon L \to \emptyset$ and a class $\phi \in \mathsf{Kh}(L)$, the integer

$$\Sigma_\phi := |\mathsf{Kh}(\Sigma)(\phi)| \in \mathbb{Z}$$

is an invariant of the boundary-preserving isotopy class of Σ.

Questions:
Do these invariants distinguish any surfaces?
Are they better than Khovanov-Jacobsson classes?

Quick results

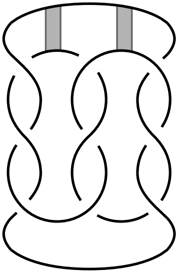
Theorem (Hayden-S. '21)

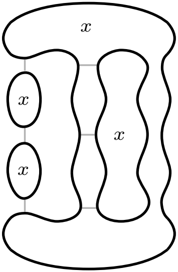
The pair of slice disks D_ℓ and D_r for the knot K (below) induce distinct maps on Khovanov homology, distinguished by the given class $\phi \in \mathsf{Kh}(K)$, and therefore, are not isotopic rel boundary.

Quick results

Theorem (Hayden-S. '21)

The pair of slice disks D_ℓ and D_r for the knot K (below) induce distinct maps on Khovanov homology, distinguished by the given class $\phi \in \mathsf{Kh}(K)$, and therefore, are not isotopic rel boundary.

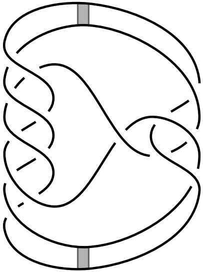




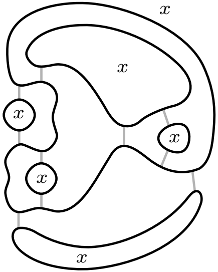
Quick results

Theorem (Hayden-S. '21)

The pair of slice disks D_ℓ and D_r for the knot K (below) induce distinct maps on Khovanov homology, distinguished by the given class $\phi \in \mathsf{Kh}(K)$, and therefore, are not isotopic rel boundary.



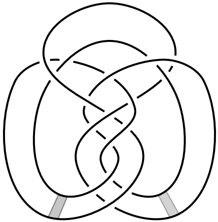
$15n_{103488}$



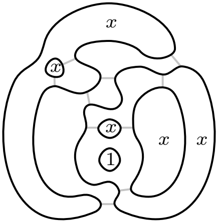
Quick results

Theorem (Hayden-S. '21)

The pair of slice disks D_ℓ and D_r for the knot K (below) induce distinct maps on Khovanov homology, distinguished by the given class $\phi \in \mathsf{Kh}(K)$, and therefore, are not isotopic rel boundary.



$17nh_{74}$



Exotic slices

Fact:
The slices for 6_1, 9_{46}, and $15n_{103488}$ are not even topologically isotopic rel boundary.

Exotic slices

Fact:
The slices for 6_1, 9_{46}, and $15n_{103488}$ are not even topologically isotopic rel boundary.

Definition

A pair of slice disks are *exotic* if they are topologically isotopic rel boundary, but not smoothly isotopic rel boundary.

Exotic slices

Fact:
The slices for 6_1, 9_{46}, and $15n_{103488}$ are not even topologically isotopic rel boundary.

Definition

A pair of slice disks are *exotic* if they are topologically isotopic rel boundary, but not smoothly isotopic rel boundary.

Theorem (Hayden '21)

The slices for $17nh_{74}$ are topologically isotopic rel boundary.

Exotic slices

Fact:
The slices for 6_1, 9_{46}, and $15n_{103488}$ are not even topologically isotopic rel boundary.

Definition

A pair of slice disks are *exotic* if they are topologically isotopic rel boundary, but not smoothly isotopic rel boundary.

Theorem (Hayden '21)

The slices for $17nh_{74}$ are topologically isotopic rel boundary.

Corollary (Hayden-S. '21)

The induced maps on Khovanov homology detect exotic pairs of slice disks.

Exotic slices

Fact:
The slices for 6_1, 9_{46}, and $15n_{103488}$ are not even topologically isotopic rel boundary.

Definition

A pair of slice disks are *exotic* if they are topologically isotopic rel boundary, but not smoothly isotopic rel boundary.

Theorem (Hayden '21)

The slices for $17nh_{74}$ are topologically isotopic rel boundary.

Corollary (Hayden-S. '21)

The induced maps on Khovanov homology detect exotic pairs of slice disks.

Can be extended to an infinite family of knots bounding pairs of ambiently non-isotopic surfaces of any genus.

Comparisons

Case 1:

- It is hard to compute KJ_Σ
- It is hard to compare KJ_Σ and $\mathrm{KJ}_{\Sigma'}$

Comparisons

Case 1:

- It is hard to compute KJ_Σ
- It is hard to compare KJ_Σ and $\mathsf{KJ}_{\Sigma'}$

Case 2:

- By choosing ϕ wisely, it is easier to compute Σ_ϕ
- Comparing integers is easy

Table of Contents

1 Motivation

2 Khovanov homology of surfaces

3 Khovanov homology of knotted surfaces

4 Khovanov homology of slice disks

5 Future work

Future work

- tweak the algebra (e.g. annular Khovanov homology)

Future work

- tweak the algebra (e.g. annular Khovanov homology)
- tweak the topology (slice disks in different 4-manifolds)

Future work

- tweak the algebra (e.g. annular Khovanov homology)
- tweak the topology (slice disks in different 4-manifolds)
- study different families of disks

Future work

- tweak the algebra (e.g. annular Khovanov homology)
- tweak the topology (slice disks in different 4-manifolds)
- study different families of disks
- study relationship with other invariants (e.g. s-invariant or knot Floer homology)

Future work

- tweak the algebra (e.g. annular Khovanov homology)
- tweak the topology (slice disks in different 4-manifolds)
- study different families of disks
- study relationship with other invariants (e.g. s-invariant or knot Floer homology)
- study slice obstruction from Khovanov-Jacobsson classes

Thank You!

Thank you!

Bibliography I

- D Bar-Natan, *Khovanov's homology for tangles and cobordisms*, **Geom. Topol.**, 9:1443-1499, 2005.
- *Characterisation of homotopy ribbon discs*, **Adv. Math.**, 391:Paper No. 107960, 2021.
- Kyle Hayden, *Corks, covers, and complex curves*, arXiv:2107.06856, 2021.
- Kyle Hayden and Isaac Sundberg, *Khovanov homology and exotic surfaces in the 4-ball*, arXiv:2108.04810, 2021.
- Magnus Jacobsson, *An invariant of link cobordisms from Khovanov homology*, **Algebr. Geom. Topol.**, 4:1211-1251, 2004.
- András Juhász and Ian Zemke, *Distinguishing slice disks using knot floer homology*, *Seceta Math. (N.S.)*,20(1), 2020.
- Mikhail Khovanov, *A categorification of the Jones polynomial*, **Duke Math. J.**, 101(3):359-426, 2000.
- Mikhail Khovanov, *An invariant of tangle cobordisms*, **Transactions of the American Mathematical Society**, 358(1):315-327, 2006.

Bibliography II

- Adam Simon Levine and Ian Zemke, *Khovanov homology and ribbon concordances*, **Bull. Lond. Math. Soc.**, 51(6):1099-1103, 2019.
- Allison N. Miller and Mark Powell, Stabilization distance between surfaces, *Enseign. Math.*,**65**:397-440, 2020.
- Lisa Piccirillo, *The Conway knot is not slice*, **Ann. of Math.** (2), 191(2):581-591, 2020.
- Jacob Rasmussen, *Khovanov's invariant for closed surfaces*, arXiv:math/0502527, 2005.
- Isaac Sundberg and Jonah Swann, *Relative Khovanov-Jacobsson classes*, arXiv:2103.01438, 2021.
- Jonah Swann, *Relative Khovanov-Jacobsson classes of spanning surfaces*, *Ph.D. Thesis, Bryn Mawr College*, 2010.
- Kokoro Tanaka, *Khovanov-Jacobsson numbers and invariants of surface-knots derived from Bar-Natan's theory*, **Proc. Amer. Math. Soc.**, 134(12):3685–3689, 2005.