Khovanov homology and uniqueness of surfaces in the 4-ball

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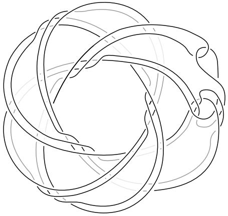
DMV-Jahrestagung

16 September 2022



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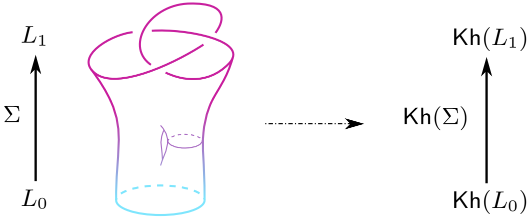

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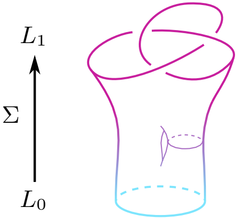			

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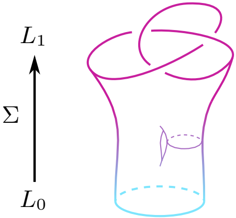			

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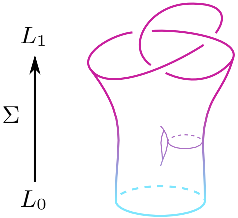
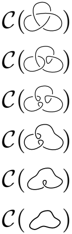
<tr><td>
</td><td></td><td></td><td></td></tr>
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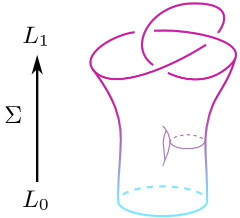		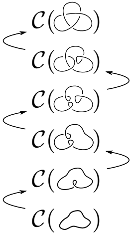	

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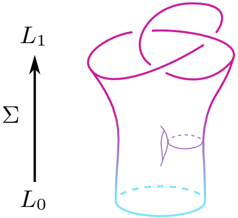		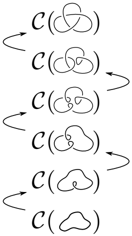	$\mathcal{C}(D_1)$ ↑ $\mathcal{C}(\Sigma)$ $\mathcal{C}(D_0)$

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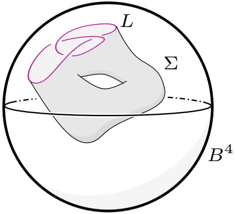
Theorem (MWW19)

Invariance of $\mathrm{Kh}(\Sigma)$ *holds for link cobordisms in* $S^3 \times [0,1]$.

Surfaces in the 4-ball



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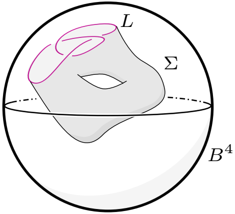

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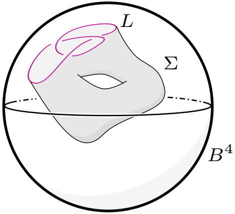

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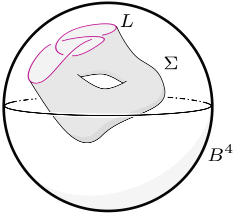

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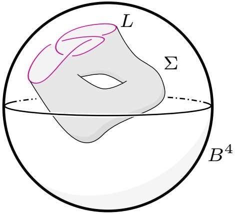

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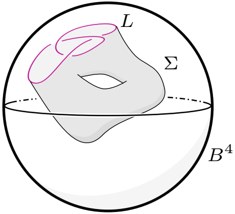

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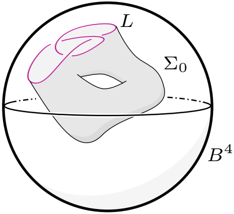

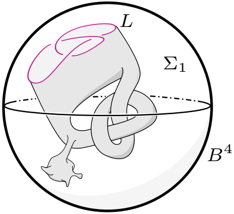

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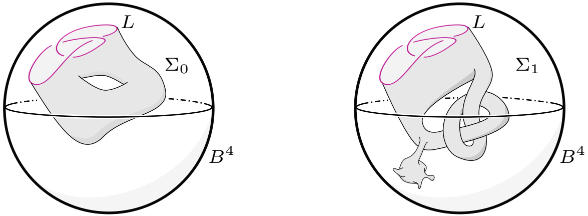

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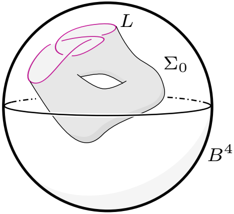

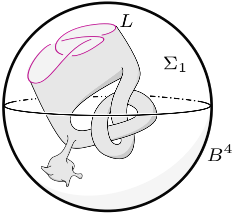

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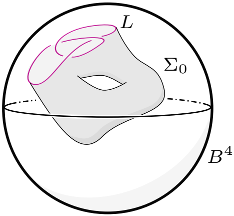

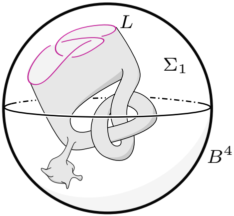

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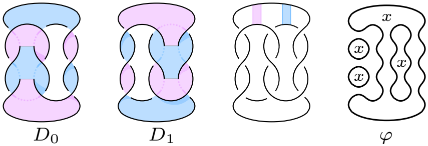

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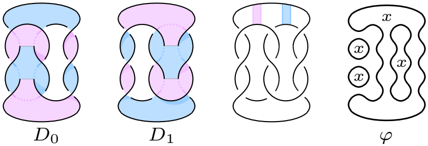

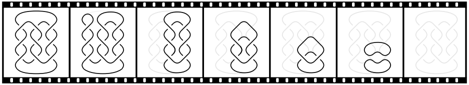

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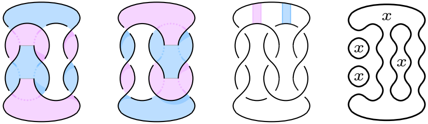
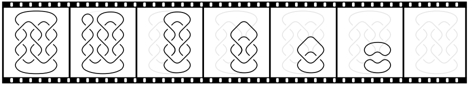
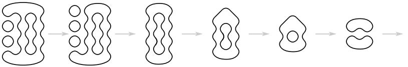


D_0 D_1 φ





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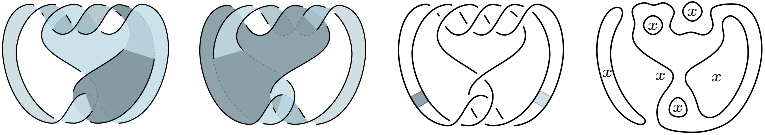

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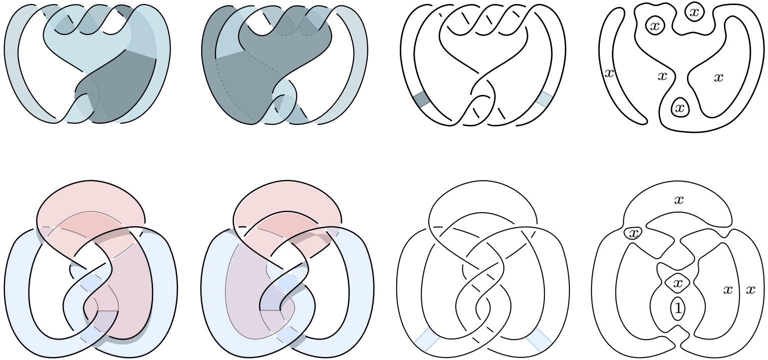

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- recent work extends to nonorientable surfaces [LS21]

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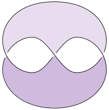

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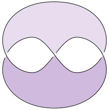

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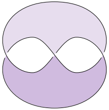

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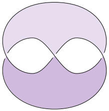

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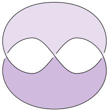

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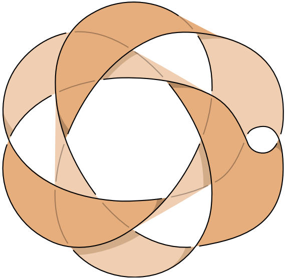

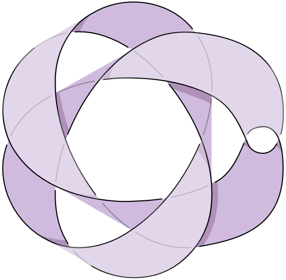

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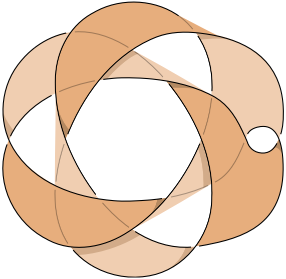

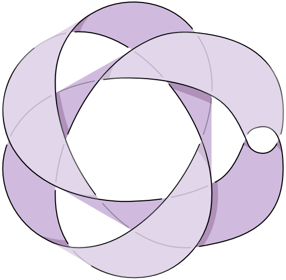

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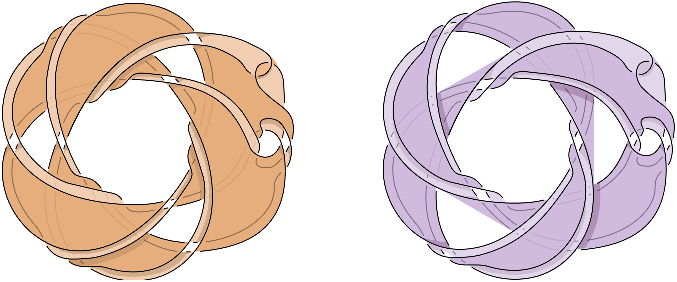

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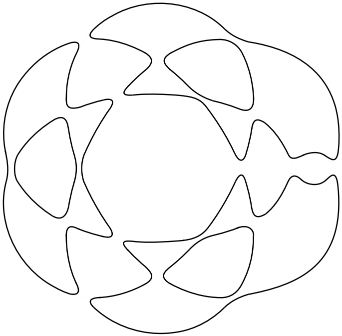

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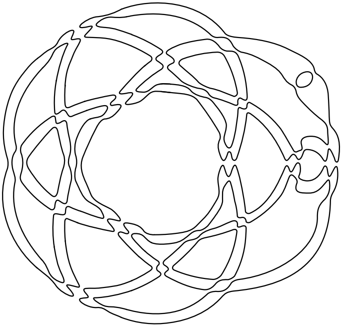

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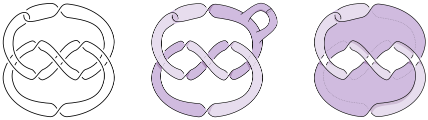

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